Diplomarbeit

Linearsysteme auf (Hilbert-Schemata von) K3 Flächen
Linear systems on (Hilbert schemes of) K3 surfaces

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Einleitung

> Sie sind so schön, diese Gleichungen. Sogar optisch schön, aber besonders schön im geistigen Sinne. Ihre Präzision und ihre Kraft sind schön, und wenn er beginnt, eine Gleichung zu verstehen, bekommt er dasselbe Gefühl, wie wenn er den Mond über den Bäumen aufgehen sieht. In seiner Seele ist es dunkel und still, und dann beginnen die Baumwipfel auf der gegenüberliegenden Seite der Bucht ein wenig zu glühen, weiß und sanft, und das Weiß wird heller, läßt die Umrisse der Bäume hervortreten, und schließlich wird ein kleines Stück des Mondes sichtbar, und die Mathematik tut sich auf, umfaßt das alles und scheint in Vollendung.
>
> *(Alan Lightman - Der gute Benito)*

In dieser Arbeit geht es um Einbettungen von projektiven Varietäten in projektive Räume. Morphismen in den projektiven Raum sind bestimmt durch ein Linienbündel und eine Menge von globalen Schnitten. Jede dieser Einbettungen liefert umgekehrt auch ein Linienbündel und eine Menge von globalen Schnitten. Es besteht ein enger Zusammenhang zwischen Bündeln zusammen mit globalen Schnitten einerseits und den Einbettungen andererseits. So ist die Einbettung genau dann projektiv normal, das heißt der homogene Koordinatenring von dem Bild der Varietät ganz abgeschlossen, wenn das Linienbündel normal erzeugt ist, also die natürlichen Abbildungen $S^n(H^0(L)) \to H^0(L^n)$ surjektiv sind. Im Lichte einer bestimmten freien Auflösung des Ringes $\oplus_n H^0(L^n)$ betrachtet, erkennt man im ersten Term dieser Auflösung die Bedingung, dass das Linienbündel normal erzeugt ist. Hieraus ergibt sich natürlich die Frage, wann die nächsten Terme so einfach wie möglich sind. Das führt zur Definition der Eigenschaft N_p. Für $p = 0$ ist das genau die Eigenschaft „normal erzeugt“. Für $p = 1$ bedeutet N_p, dass das Ideal, das zu der Varietät gehört, quadratisch erzeugt ist.
Im Falle einer K3 Fläche gibt es hierzu Ergebnisse von Saint-Donat aus dem Jahre 1974: In seiner berühmten Arbeit [SD74] zeigt er unter anderem, dass für jedes ample Linienbündel L auf einer K3-Fläche, L^2 normal erzeugt ist (siehe Theorem 6.1 (ii) in seiner Arbeit). Darüberhinaus zeigt er, dass das L^2 die Eigenschaft N_1 erfüllt, also das Ideal quadratisch erzeugt ist (siehe im wesentlichen Theorem 7.2).
In einer anderen Herangehensweise als der von Saint-Donat nutzt man den Kern der Auswertungsabbildung M_L. Auf diese Weise lässt sich zeigen, dass die Kohomologiegruppen $H^1(M_L^{\otimes p+1} \otimes L^s)$ für alle $s \geq p+1$ genau dann verschwinden, wenn L die Eigenschaft N_p hat (siehe 3.4). Dies benutzten Gallego und Purnapranja, um die Eigenschaft N_p für die Potenzen eines Linienbündels L auf einer K3 Fläche mit $(L.L) \geq 4$ zu untersuchen. Wir werden diese Ergebnisse im Abschnitt „Results for K3 surfaces“ vorstellen. Dabei wird die Eigenschaft, dass M_L semistabil ist, eine zentrale Rolle spielen.

In dieser Arbeit untersuchen wir Linienbündel auf verschiedenen geometrischen Objekten: Wenn S eine K3 Fläche ist und L ein Linienbündel auf ihr, betrachten wir zunächst das Bündel $L \boxtimes L$ auf dem Produkt $S \times S$. Wir zeigen, dass das Quadrat dieses Bündels normal erzeugt ist und die $p+1$-te Potenz Eigenschaft N_p hat. Danach untersuchen wir das Verhalten unter Aufblasungen. Wenn ein Bündel Eigenschaft N_p hat, so hat auch das auf die Aufblasung zurückgezogene Bündel diese Eigenschaft. Als Letztes gehen wir auf das symmetrische Produkt bzw. auf das Hilbert Schema ein und zeigen dort, dass ein normal erzeugtes Bündel beim Abstieg auch normal erzeugt bleibt.
Die Arbeit ist wie folgt aufgebaut: Nach der Einleitung im ersten Kapitel führen wir im zweiten Kapitel die für die Arbeit nötigen Bezeichnungen ein. Anschließend widmen wir uns dem Hilbert Schema: Wir motivieren die Konstruktion und definieren es. Die grundlegenden Eigenschaften dieses Schemas werden erklärt. Dem Kern der Auswertungsabbildung ist das dritte Kapitel gewidmet. Dabei wird auch Eigenschaft N_p eingeführt. Danach werden die Ergebnisse von Gallego und Purnapranja vorgestellt und bewiesen. Die Idee dahinter ist, die Aussagen auf eine Kurve zurückzuführen, auf der man mittels Stabilitätsaussagen die Kohomologiegruppen kontrolliert. Wir berechnen die höheren Kohomologiegruppen und beweisen zwei für die das vierte Kapitel relevante Lemmata. Im Abschnitt „Properties of M_G“ entwickeln wir Formeln, die das Verhalten von M_L beschreiben: Wir drücken $M_{L \boxtimes L}$ in Termen von M_L aus und zeigen, dass das Zurückziehen auf eine Aufblasung mit $M_{(-)}$ vertauscht. Im vierten Kapitel nutzen wir das Kriterium von verschwindenden Kohomologien, um Eigenschaft N_p bzw. das Normal-Erzeugt-Sein zu zeigen. Die Abschnitte sind ähnlich aufgebaut: Unter bestimmten Umständen verschwindet die Gruppe $H^1(M_{L^r} \otimes L^s)$ für bestimmte r und s, womit L^r normal erzeugt bzw. die Eigenschaft N_p erfüllt. Die Idee dahinter ist, die Kohomologiegruppe $H^1(M_{L^r} \otimes L^s)$ mit Hilfe der Aussagen aus dem dritten Kapitel auf etwas Bekanntes zurückzuführen und so zum Verschwinden zu bringen. Daraus folgen direkt die gewünschten Eigenschaften des Linienbündels.

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Contents

Chapter 1

Introduction

Rien n'est beau que le vrai.

(Hermann Minkowski)

This thesis deals with embeddings of projective varieties into projective spaces. Morphisms into projective space are determined by a line bundle and a set of global sections. Conversely, every such embedding gives us a line bundle and a set of global sections. There is a close relationship between this line bundle and the set of sections on the one hand and the embedding on the other. For example: The embedding is projectively normal, i.e. the homogeneous coordinate ring of the variety's image is integrally closed if and only if the line bundle is normally generated, i.e. the canonical mappings $S^n(H^0(L)) \to H^0(L^n)$ are onto. One can view this condition on a line bundle being normally generated in the light of a certain free resolution of $\oplus_n H^0(L^n)$. From the first term of the resolution, we can deduce the condition that L has to be normally generated. To us, it seems to be an obvious question, under which circumstances the next terms are as simple as possible. This leads us to the definition of N_p. For $p = 0$ this is the property of being "normally generated". For $p = 1$ this means that the varieties ideal is quadratic, i.e. generated by quadratic polynomials.
In the case of a K3 surface there are results by Saint-Donat: In his well known article [SD74] he proved the following: If L is an ample line bundle on a K3 surface, L^2 is normally generated (Theorem 6.1 (ii) in his article). Furthermore, he shows that in this case, the ideal has to be quadratic (check Theorem 7.2 of Saint-Donat), i.e. L^2 satisfies the property N_1.
An approach different from Saint-Donat's is using the kernel M_L of the evaluation map. One can show that the cohomology group $H^1(M_L^{\otimes p+1} \otimes L^s)$ vanishes for all $s \geq p+1$ if and only if L satisfies N_p (check 3.4). Gallego and Purnaprajna used this to prove the property N_p for L^r, given that L is a line bundle with $(L.L) \geq 4$ on a K3 surface. We will present these results in section "Results for K3 surfaces". The fact that M_L is semistable will be a central tool for this.
We consider line bundles on different geometrical objects: If S is a K3 surface and L a line bundle on S, we look at the bundle $L \boxtimes L$ on the product $S \times S$. We show that the square of this bundle is normally generated and the $(p+1)$-th power has the property N_p. Consequently, we focus on the behavior under blowup. If a bundle has the property N_p, then so does the pullback to the blowup. Eventually, we consider the symmetric product and the Hilbert scheme, and show that a normally generated bundle remains normally

generated under descent.
This thesis is structured as follows:
In chapter two we will introduce the required notation. Then, we present the Hilbert scheme, motivating the construction and giving the definition. We illustrate the basic properties of this scheme.
The kernel of the evaluation map is the topic of the third chapter. Alongside, we introduce the property N_p. The results of Gallego and Purnaprajna will be stated and proved. The basic idea behind this is to reduce the statement to a curve and handle the cohomology groups with stability theorems. We calculate higher cohomology groups and prove two technical lemmas for chapter three. We develop formulas that characterize the behavior of M_L in section "Properties of M_G": Expressing $M_{L\boxtimes L}$ in terms of M_L, we show that the pullback under blowups commutes with $M_{(-)}$.
In chapter four, we use the vanishing cohomology criterion to verify property N_p and the normal generation. The sections are structured in a similar fashion: The group $H^1(M_{L^r} \otimes L^s)$ vanishes under certain circumstances for certain r and s – thereby, L^r is normally generated, respectively satisfies N_p. The basic idea behind this is to reduce the cohomology group $H^1(M_{L^r} \otimes L^s)$ to something well-known that vanishes – this is achieved using the statements of chapter three. The desired properties of the line bundle follow from this.

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In the German acknowledgment one important person is missing. I would like to give a special thank to Jesko Hüttenhein for proofreading my thesis and thereby, teaching me some English.

Chapter 2

General notations and the Hilbert Scheme

Some general remarks and notations

Throughout this paper, we will work over the complex numbers $\mathbb{C}$.
Given the product of two varieties, we denote the projection to the first and the second component by π_1 and π_2. The line bundle $\pi_1^*L \otimes \pi_2^*L'$ for L and L' line bundles on the first and second variety, respectively, will sometimes be written as $L \boxtimes L'$. The self-intersection number of L is denoted by $(L.L)$ and its tensor powers with $L \otimes \cdots \otimes L$ (r-times) by L^r .
The n-th symmetric power of a projective variety X is written as $S^n(X)$ and obtained in the following way: Let $\mathfrak{S}_n$ be the symmetric group on n letters. It operates on X^n by permuting the components. The quotient $S^n(X) := X^n/\mathfrak{S}_n$ exists as a projective variety, since $\mathfrak{S}_n$ is a finite group and we can apply [BF05, Theorem 7.1.2].
In the course of this thesis we will need the sum and the intersection of subsheaves of a sheaf of modules. If L is a product, we will be able to express the bundle M_L in terms of the M_{L_i} where the L_i are the "components" of L (see Theorem 3.20).

Definition. Let $(X, \mathcal{O}_X)$ be a ringed space, $\mathscr{F}$ an $\mathcal{O}_X$-module and $\mathscr{G}$, $\mathscr{G}'$ two $\mathcal{O}_X$-submodules of $\mathscr{F}$. Then we define the $\mathcal{O}_X$-modules:

(i) $\mathscr{G} + \mathscr{G}'$: $(\mathscr{G} + \mathscr{G}')(U) = \mathscr{G}(U) + \mathscr{G}'(U)$ for all open $U \subset X$.

(ii) $\mathscr{G} \cap \mathscr{G}'$: $(\mathscr{G} \cap \mathscr{G}')(U) = \mathscr{G}(U) \cap \mathscr{G}'(U)$ for all open $U \subset X$.

The presheaves $\mathscr{G} + \mathscr{G}'$ and $\mathscr{G} \cap \mathscr{G}'$ actually are sheaves, since $\mathscr{G}(U)$ and $\mathscr{G}'(U)$ are $\mathscr{F}(U)$-submodules for all U.
As in the local case, we get an exact sequence of $\mathcal{O}_X$-modules:

$$0 \longrightarrow \mathscr{G} \cap \mathscr{G}' \longrightarrow \mathscr{G} \oplus \mathscr{G}' \longrightarrow \mathscr{G} + \mathscr{G}' \longrightarrow 0.$$

The exactness of this sequence can be checked on the stalks. The induced long exact sequence helps us to show that the cohomology groups of the sum vanish. If $H^i(\mathscr{G})$, $H^i(\mathscr{G}')$ and $H^{i+1}(\mathscr{G} \cap \mathscr{G}')$ vanish, then $H^i(\mathscr{G} + \mathscr{G}')$ vanishes as well:

$$\cdots \longrightarrow H^i(\mathscr{G} \oplus \mathscr{G}') = H^i(\mathscr{G}) \oplus H^i(\mathscr{G}') \longrightarrow H^i(\mathscr{G} + \mathscr{G}') \longrightarrow H^{i+1}(\mathscr{G} \cap \mathscr{G}') \longrightarrow \cdots . \tag{2.1}$$

Of course we can look at more than two summands. Then, we inductively obtain the following

2.2 Lemma. *Let $(X, \mathcal{O}_X)$ be a ringed space, $\mathscr{F}$ an $\mathcal{O}_X$-module and $\mathscr{G}_1, \dots, \mathscr{G}_n$ $\mathcal{O}_X$-submodules of $\mathscr{F}$. If for all non-empty subsets $\Lambda \subseteq \{1, \dots, n\}$ the cohomology group $H^{i-1+|\Lambda|}(\bigcap_{l\in\Lambda} \mathscr{G}_l)$ vanishes, then $H^i(\sum_{j=1}^n \mathscr{G}_j)$ vanishes as well. Here $|\Lambda|$ denotes the cardinality of Λ.*

Proof. We prove this lemma by induction on n. It is trivial for a single summand: The desired vanishing is the same as in the assumption.
Hence, assume the vanishing for n summands. We want to show $0 = H^i(\sum_{j=1}^{n+1} \mathscr{G}_j) = H^i(\sum_{j=1}^n \mathscr{G}_j + \mathscr{G}_{n+1})$. By the sequence (2.1), we want three groups to vanish:

(i) $H^i(\mathscr{G}_{n+1}) = 0$:
Vanishes by the assumption; take $\Lambda = \{n + 1\}$.

(ii) $H^i(\sum_{j=1}^n \mathscr{G}_j) = 0$:
Since for all $\Lambda \subseteq \{1, \dots, n\} \subseteq \{1, \dots, n, n + 1\}$ we have $H^{i-1+|\Lambda|}(\bigcap_{j\in\Lambda} \mathscr{G}_j) = 0$. The induction gives us $H^i(\sum_{j=1}^n \mathscr{G}_j) = 0$.

(iii) $H^{i+1}(\sum_{j=1}^n (\mathscr{G}_j \cap \mathscr{G}_{n+1})) = 0$:
Again we want to use the induction hypothesis. We have to check that, for all $\Lambda \subset \{1, \dots, n\}$, we have $H^{i+|\Lambda|}(\bigcap_{j\in\Lambda} \mathscr{G}_j \cap \mathscr{G}_{n+1}) = 0$.
Define $\Delta := \Lambda \cup \{n + 1\}$. Thus $H^{i+|\Lambda|}(\bigcap_{j\in\Lambda} \mathscr{G}_j \cap \mathscr{G}_{n+1}) = H^{i-1+|\Delta|}(\bigcap_{j\in\Delta} \mathscr{G}_j) = 0$ by induction.

q.e.d.

Using the notion of the sum of subsheaves, we can describe the kernel of a morphism given as a tensor product of morphisms.
Let $f : \mathscr{F} \to \mathscr{G}$ and $g : \mathscr{F}' \to \mathscr{G}'$ be two morphisms of locally free $\mathcal{O}_X$-modules. Then, the map $f \otimes g : \mathscr{F} \otimes \mathscr{F}' \to \mathscr{G} \otimes \mathscr{G}'$ has kernel

$$\ker(f \otimes g) = \ker(f) \otimes \mathscr{F}' + \mathscr{F} \otimes \ker(g). \tag{2.3}$$

The equation can be checked on the stalks: For all $x \in X$ $\mathscr{F}_x, \mathscr{F}'_x, \mathscr{G}_x$ and $\mathscr{G}'_x$ are free $\mathcal{O}_{X,x}$-modules. In this case, (2.3) can be deduced by basic calculation.
It will be useful to know the (higher) direct images of the structure sheaf under blowup, because later on we will construct the Hilbert scheme as a blowup.

2.4 Theorem. *Let X be a smooth variety and $\varphi : \widetilde{X} \to X$ be the blowup along some subvariety. Then:*

$$\varphi_*\mathcal{O}_{\widetilde{X}} = \mathcal{O}_X \text{ and } R^i\varphi_*\mathcal{O}_{\widetilde{X}} = 0.$$

Proof. The map φ is a surjective, proper and birational morphism of smooth varieties. Thus we can apply [Vie77, Lemma 1]. q.e.d.

The Hilbert scheme

In the focus of our interests lay the symmetric power and the Hilbert scheme of points on a K3 surface. The symmetric power had an intuitive definition. In contrast, the Hilbert scheme requires a more abstract approach. One can think of the Hilbert scheme as the parameter space for subschemes of a given scheme. This scheme is the disjoint union of several projective schemes, each of them corresponding to a Hilbert polynomial. In other words, it parametrizes all subschemes with a given Hilbert Polynomial. This scheme, first defined by Grothendieck in [Gro61], is a fundamental object in algebraic geometry: Together with the Quot Scheme, it plays a central role in the construction of moduli spaces. For example the moduli space of polarized K3 surfaces[1] is constructed as a quotient of a subscheme of the Hilbert scheme.
The Hilbert scheme (of the projective space) itself was constructed by Grothendieck as a vanishing set of certain determinants in the Grassmannian.
In this thesis we do not want to use the Hilbert scheme to perform any constructions, we are rather interested in the scheme itself. In particular, we lay our focus on the constant Hilbert Polynomial n, for n a natural number. This can be seen as the parameter space for n points in a variety. Later on we will additionally assume that $n = 2$ and the points lay on a K3 surface. For the definition and the description of the properties we mainly take the notes of Lehn [Leh].

Definition. Let X be a smooth, quasi-projective scheme. We define the *Hilbert scheme* of n points on X as

$$\mathrm{Hilb}^n(X) = X^{[n]} = \{Z \subset X \,|\, \dim(Z) = 0,\ \dim(H^0(\mathcal{O}_Z)) = n\}.$$

Equivalently, one can say that the functor

$$\begin{aligned} &\mathrm{Hilb}^n(X) : \big(\mathrm{Schemes}\big)^{\mathrm{op}} \longrightarrow \big(\mathrm{Sets}\big) \\ &\mathrm{Hilb}^n(X)(S) = \{Z \subseteq S \times X | Z \text{ is proper and flat over } S, P(Z_s) = n\ \forall s \in S\} \end{aligned}$$

is represented by a scheme $\mathrm{Hilb}^n(X)$. The Hilbert polynomial of Z_s is denoted with $P(Z_s)$.

It seems a natural question to ask how the geometry of $\mathrm{Hilb}^n(X)$ is related to the geometry of X. If X is a surface, the Hilbert scheme behaves well. Therefore, we restrict to the case of a surface from now on.

2.5 Lemma. *If S is a smooth, connected, quasi-projective surface, then* $\mathrm{Hilb}^n(S)$ *is smooth and connected of dimension* $2n$ *for all* $n \in \mathbb{N}$.

This lemma (and the following ones) are proved for example in [Leh], [BF05] or [Fog68]. The intuitive conception of the Hilbert scheme as a parameter space of n points suggests

[1] K3 surfaces with an ample, indivisible line bundle.

a connection to the symmetric product. This connection is provided by the so called *Hilbert-Chow morphism.* Although it is defined only set-theoretical, one can show that it actually is a morphism.

$$\begin{aligned}\rho : \mathrm{Hilb}^n(S) &\to S^n(S)\\ [Z] &\mapsto \sum_{x\in S} l(\mathcal{O}_{Z,x})x\end{aligned}$$

Here $l(\mathcal{O}_{Z,x})$ denotes the length of $\mathcal{O}_{Z,x}$.
For a smooth curve, this morphism actually is an isomorphism. For an arbitrary S, it is still an isomorphism over the open subset of $S^n(S)$ corresponding to the n-tuples of distinct points.
Working with the symmetric product has the disadvantage that one has to deal with singular varieties. Using the Hilbert scheme these difficulties can be avoided.

2.6 Lemma. *If S is a smooth surface, then for all $n \geq 0$ the Hilbert scheme $\mathrm{Hilb}^n(S)$ is smooth. In particular, the Hilbert-Chow morphism $\rho : \mathrm{Hilb}^n(S) \to S^n(S)$ is a resolution of the singularities of the symmetric product.*

Emphasizing the special case $n = 2$, we know that $S^2(S)$ is singular exactly along the diagonal Δ and therefore, the lemma yields that $\mathrm{Hilb}^2(S)$ is the blowup of $S^2(S)$ along the image of the diagonal and the blowup morphism is the Hilbert-Chow morphism.
We also can go the other way: First we blow up $S \times S$ along the diagonal and then divide out by the induced operation of $\mathfrak{S}_2$. The result is again the Hilbert scheme. The involved maps (blowups and projections) commute.
Altogether, we can express it by the following commutative diagram:

$$\begin{tikzcd} \widetilde{S\times S} \arrow[r, "\varphi"] \arrow[d, "\widetilde{p}"'] & S\times S \arrow[d, "p"] & \Delta \arrow[l, hook'] \arrow[d, "p|_\Delta"'] \\ \mathrm{Hilb}^2(S) \arrow[r, "\bar{\varphi}"'] & S^2(S) & p(\Delta) \arrow[l, hook'] \end{tikzcd} \tag{2.7}$$

The map φ (resp. $\bar{\varphi}$) is the blowup map and p (resp. $\widetilde{p}$) the projection.
The maps p and $\widetilde{p}$ are branched double covers, so the direct images of the structure sheaves in the top row can be written as:

$$p_*\mathcal{O}_{S^2} = \mathcal{O}_{S^2(S)} \oplus I \quad R^ip_*\mathcal{O}_{S^2} = 0 \ \ \forall i \geq 1. \tag{2.8}$$

$$\widetilde{p}_*\mathcal{O}_{\widetilde{S\times S}} = \mathcal{O}_{\mathrm{Hilb}^2(S)} \oplus \widetilde{I} \quad R^i\widetilde{p}_*\mathcal{O}_{\widetilde{S\times S}} = 0 \ \ \forall i \geq 1. \tag{2.9}$$

for some line bundle I on $S^2(S)$ and $\widetilde{I}$ on the Hilbert scheme.
Now we want to relate the cohomology (of the structure sheaves) for $S^n(S)$ and $\mathrm{Hilb}^n(S)$ to the cohomology of S. The Künneth formula $H^*(S^n, \mathcal{O}_{S^n}) \simeq H^*(S, \mathcal{O}_S)^{\otimes n}$ induces an isomorphism

$$H^*(S^n(S), \mathcal{O}_{S^n(S)}) \simeq S^n(H^*(S, \mathcal{O}_S)).$$

Since the Hilbert-Chow morphism is a resolution of the singularities of $S^n(S)$ (see Lemma 2.6), its higher direct images vanish, $R^i\rho_*\mathcal{O}_{\mathrm{Hilb}^n(S)} = 0$ for all $i > 0$. Now the Leray spectral sequence yields $H^*(\mathrm{Hilb}^n(S), \mathcal{O}_{\mathrm{Hilb}^n(S)}) = H^*(S^n(S), \mathcal{O}_{S^n(S)})$.
Putting these two isomorphisms together, we get

$$H^*(\mathrm{Hilb}^n(S), \mathcal{O}_{\mathrm{Hilb}^n(S)}) = S^n H^*(S, \mathcal{O}_S).$$

With these formulas, the cohomology can be calculated if $H^*(S)$ is known.
As an example, we know the cohomology of a K3 surface. This is the case we are interested in. Here we have:
$H^0(S, \mathcal{O}_S) = H^2(S, \mathcal{O}_S) = \mathbb{C}$ and $H^i(S, \mathcal{O}_S) = 0$ for all $i \neq 0, 2$.
Hence, we get:

$$H^i(S^n(S), \mathcal{O}_{S^n(S)}) = H^i(\mathrm{Hilb}^n(S), \mathcal{O}_{\mathrm{Hilb}^n(S)}) = \begin{cases} \mathbb{C} \text{ for } i \equiv 0 \pmod 2 \text{ and } i \leq 2n \\ 0 \text{ else.} \end{cases} \tag{2.10}$$

In this thesis we want to analyze special line bundles on such a Hilbert scheme, namely those that come from S.
Let L be a line bundle on S and $J = L \boxtimes L$ on $S \times S$. Then we can take the inverse image under the blowup (along Δ) of the bundle J. The symmetric group operates on this bundle by permuting the components, implying descent to the Hilbert scheme. The bundles $\mathcal{J}$ on the Hilbert scheme which we will study satisfy the property $\widetilde{p}^*\mathcal{J} = \varphi^*(\pi_1^*L \otimes \pi_2^*L)$ for a line bundle L on S and S a K3 surface, of course.

Chapter 3

The kernel of the evaluation map

Our goal is to show that certain line bundles (or their tensor powers) on the varieties described in the previous chapter are normally generated. In order to do so, we use a cohomological criterion, which involves the kernel of the evaluation map (cf. Theorems 3.2 and 3.4). In this chapter we define this kernel and explain its relation to the normal generation. By viewing the normal generation in the light of a resolution of a certain ring, it seems natural to generalize this property by putting more conditions on the resolution. The resulting property will be called "Property N_p". In particular, we will discuss its relation to cohomology.
With the help of these criterions Gallego and Purnaprajna have shown some strong vanishing theorems for line bundles on K3 surfaces in [GP00]. For example, these imply the normal generation of the second tensor power under the assumption that the self-intersection number is larger or equal to four. We will state and prove these theorems. For later use, we generalize them to higher cohomology groups. Since we explicitly want to see the normal generation, we develop some techniques to help us with the calculations of the required cohomology groups: Behavior under tensor products and blowups.

Definition of M_G

Definition. Let X be a projective variety and G a globally generated line bundle on X. Then, there is a natural surjective evaluation map of vector bundles:

$$ev_G : H^0(G) \otimes \mathcal{O}_X \to G.$$

We define the kernel of this map as M_G. It actually is a vector bundle, since G is globally generated. By construction, one has the following exact sequence:

$$0 \longrightarrow M_G \longrightarrow H^0(G) \otimes \mathcal{O}_X \xrightarrow{ev_G} G \longrightarrow 0. \tag{3.1}$$

This vector bundle controls the normal generation of a globally generated line bundle, so we recall the

Definition. Let L be a globally generated line bundle on a projective variety X. We say that L is *normally generated* if the natural maps $S^n H^0(X, L) \to H^0(X, L^n)$ are surjective for all $n \geq 1$.

Since the line bundle L is assumed to be globally generated, we get a morphism $\varphi_L : X \to \mathbb{P}(H^0(L)^*)$. This is equivalent to the question of whether X is projectively normal with respect to this morphism. By definition, this is the case if and only if its homogeneous coordinate ring is an integrally closed domain. The following theorem describes how the bundle M_G is related to the normal generation.

3.2 Theorem. *Let L be a globally generated line bundle on a projective variety X. If the cohomology group $H^1(M_L \otimes L^s)$ vanishes for all $s \geq 1$, then L is normally generated. If, in addition, $H^1(L^k) = 0$ for all $k \geq 1$, the converse is also true.*

Proof. This proof is very similar to the proof of [GP96, Lemma 1.4], but we cannot use this lemma since $H^2(\mathcal{O}_X)$ does not vanish in our case.
If we want to show that L is normally generated, we have to verify that the maps $S^n H^0(L) \to H^0(L^n)$ are surjective for all $n \geq 1$. For any n, this map fits into a commutative diagram (cf. cit. loc.):

$$\begin{array}{ccc}
H^0(L)^{\otimes n} & \longrightarrow & S^n H^0(L) \\
\downarrow{\scriptstyle \gamma_1} & & \Bigg\downarrow \\
H^0(L^2) \otimes H^0(L)^{\otimes n-2} & & \\
\downarrow{\scriptstyle \gamma_2} & & \\
\vdots & & \\
\downarrow{\scriptstyle \gamma_{n-2}} & & \\
H^0(L^{n-1}) \otimes H^0(L) & \xrightarrow{\gamma_{n-1}} & H^0(L^n)
\end{array}$$

Now we want the multiplication maps $\gamma_j : H^0(L^j) \otimes H^0(L) \to H^0(L^{j+1})$ to be surjective for all $j = 1, \ldots, n-1$. These maps can be obtained from the sequence (3.1) for $G = L$, tensored with L^j:

$$\cdots \longrightarrow H^0(L^j) \otimes H^0(L) \xrightarrow{\gamma_j} H^0(L^{j+1}) \longrightarrow H^1(M_L \otimes L^j) \longrightarrow H^0(L) \otimes H^1(L^j) \longrightarrow \cdots$$

$H^1(M_L \otimes L^j)$ vanishes, therefore γ_j is surjective.
For the converse, we additionally assume that $H^1(L^k)$ vanishes for all k. The vanishing of $H^1(M_L \otimes L^j)$ implies the surjectivity of γ_i for all i. With the above diagram, this implies the normal generation. q.e.d.

As mentioned before, L determines a morphism $\varphi_L : X \to \mathbb{P}(H^0(L)^*)$. Let $S := S^*(H^0(L)) \simeq S^*(H^0(L)^*)$ be the homogeneous coordinate ring of this projective space. We consider the graded ring $R := \bigoplus_m H^0(L^m)$, which is a finitely generated S-module in the natural way. Therefore, it has a minimal graded free resolution:

$$0 \longrightarrow E_{r-1} \longrightarrow E_{r-2} \longrightarrow \cdots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow R, \longrightarrow 0 \tag{3.3}$$

where $r = \dim H^0(L) - 1$ and each E_i is a direct sum of twists of S: $E_i = \oplus_j S(-a_{i,j})$. If we now assume L to be normally generated we get $E_0 = S$. Thus, being normally generated implies that the first term of the resolution is very simple. Consequently, if the first p terms of this resolution are "simple" in a certain sense, we wonder about the implications. Specifically, one makes the following

Definition. With the above notations we fix an integer $p \geq 0$. We say that the line bundle L on X satisfies property N_p if and only if the following hold:

- $E_0(L) = S$,
- $E_i(L) = \oplus S(-i-1) \qquad \forall 1 \leq i \leq p.$

Example. If we assume a line bundle to have property N_0, this means that we only have to satisfy the first part of the definition. However, this is the notion of a normally generated line bundle, as we defined it before.
If a line bundle is normally generated, the above resolution determines a resolution of the homogeneous ideal of X. Thus, N_1 is satisfied if and only if this ideal is generated by quadrics.

As we have seen before in Theorem 3.2, property N_0 can be checked in cohomology. In the end, we would like to come to a similar criterion for property N_p. We now give a sketch of how this property can be related to the vanishing of certain cohomology groups. If, for example, property N_1 holds, i.e. the corresponding homogeneous ideal is generated by quadrics, one looks at the group $\mathrm{Tor}_1(R, \underline{\mathbb{C}})$. Here, $\underline{\mathbb{C}}$ denotes the residue field of S at the irrelevant maximal ideal. On the one hand, this Tor group can be computed from the sequence (3.3) since it is a graded S-module. Hence, the dimension of its component in degree k is the number of minimal generators of the homogeneous ideal of degree k. We obtain that N_1 holds if and only if $\mathrm{Tor}_1(R, \underline{\mathbb{C}})$ is concentrated in the components of degree zero, one and two.
On the other hand we can compute this Tor group using the Koszul resolution of $\underline{\mathbb{C}}$:

$$0 \to S(-r-1) \otimes \bigwedge^{r+1} H^0(L) \to \cdots$$
$$\cdots \to S(-2) \otimes \bigwedge^2 H^0(L) \to S(-1) \otimes H^0(L) \to S \to \underline{\mathbb{C}} \to 0.$$

Tensoring with R and looking at the graded pieces, one finds that the k-th component of $\mathrm{Tor}_1(R, \underline{\mathbb{C}})$ is isomorphic to the homology of the complex

$$H^0(L^{k-1}) \otimes \bigwedge^2 H^0(L) \to H^0(L^k) \otimes H^0(L) \to H^0(L^{k+1}).$$

Finally, N_1 is equivalent to the exactness of this sequence, which in turn follows from the vanishing of $H^1(\bigwedge^2 M_L \otimes L^k)$ for all $k \geq 2$. If, in addition, $H^1(L^k) = 0$ for all k, the converse is also true.
As done in [Laz89] and [GL88], this proof can be generalized to property N_p and yields the following

3.4 Theorem. *Let L be a globally generated line bundle on a projective variety X. If the cohomology group $H^1(X, \bigwedge^{p'+1} M_L \otimes L^k)$ vanishes for all $0 \leq p' \leq p$ and all $k \geq 1$, then L satisfies the property N_p. If, in addition, $H^1(L^t) = 0$ for all $t \geq 1$, then the above condition is necessary and sufficient for L to satisfy property N_p.*

Since we are working over the complex numbers, it is sufficient to prove the vanishing of $H^1(M_L^{\otimes p+1} \otimes L^k)$ for all $k \geq 1$ to get property N_p.

Results for K3 surfaces

As we have seen, the normal generation of a line bundle can be checked by calculation of the cohomology groups $H^1(M_L \otimes L^k)$ for all k. Let us now use this technique to show appropriate results for line bundles on K3 surfaces. These strong vanishing theorems were stated and proved by Gallego and Purnaprajna in [GP00].

3.5 Theorem. *Let S be a K3 surface and let L be a globally generated line bundle on S with $(L.L) \geq 4$. Then the multiplication map*

$$H^0(L^r) \otimes H^0(L^s) \to H^0(L^{r+s})$$

is surjective for all $r \geq 2, s \geq 1$. Moreover, $H^1(M_{L^r} \otimes L^s) = 0$ for all $r \geq 2, s \geq 1$ and all $r \geq 1, s \geq 2$.

Before we begin with the proof of the theorem, we make a few remarks.

3.6. Since a K3 surface is a nonsingular projective variety and L is globally generated, we can apply Bertini's theorem to the base point free linear system $|L|$. Therefore almost every $C \in |L|$, considered as a closed subscheme, is smooth (cf. [Har77, III.10.9]). In particular, we get the existence of a smooth curve $C \in |L|$ and can write $L = \mathcal{O}(C)$.

3.7. For three coherent sheaves E, F and G and their multiplication maps, we can give a commutative diagram

$$\begin{tikzcd}
H^0(E) \otimes H^0(F) \otimes H^0(G) \arrow[r] \arrow[d, "m_1"'] & H^0(E \otimes F) \otimes H^0(G) \arrow[d, "m"] \\
H^0(E \otimes G) \otimes H^0(F) \arrow[r, "m_2"] & H^0(E \otimes F \otimes G).
\end{tikzcd}$$

From this diagram, it follows that surjectivity of m_1 and m_2 implies surjectivity of m.

3.8. Let X be a regular variety, Y a subvariety of codimension 1, $L := \mathcal{O}(Y)$ a line bundle on X and F a coherent sheaf such that $H^1(F \otimes L^*) = 0$. We consider Y as an effective divisor and get the sequence:

$$0 \to \mathcal{O}(-Y) \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

We twist this sequence with $\mathcal{O}(Y)$, take global sections and tensor the result with $H^0(F)$. We obtain a sequence where the terms give rise to a multiplication:

$$\begin{tikzcd}
H^0(F)\otimes H^0(\mathcal{O}_X) \arrow[r, hook] \arrow[d] & H^0(F)\otimes H^0(L) \arrow[r, two heads] \arrow[d, "\alpha"] & H^0(F)\otimes H^0(L\otimes \mathcal{O}_Y) \arrow[d, "\beta"] \\
H^0(F) \arrow[r, hook] & H^0(F\otimes L) \arrow[r, two heads] & H^0(F\otimes L\otimes \mathcal{O}_Y)
\end{tikzcd}$$

The right map in the top row is surjective due to the regularity of X. If we furthermore assume that the multiplication map on the curve is surjective, namely

$$H^0(F\otimes \mathcal{O}_Y)\otimes H^0(L\otimes \mathcal{O}_Y) \twoheadrightarrow H^0(F\otimes L\otimes \mathcal{O}_Y),$$

the map β is also surjective. But if β is onto, the multiplication map α must be surjective as well. Summing up: In the given situation, the multiplication map is surjective if it is on the curve.

3.9. In the paper [But94] of Butler, a criterion is presented to check whether the multiplication map on a curve is surjective. In our case of line bundles, the criterion can be formulated in the following way: If L and L' are line bundles on a curve of genus g such that L' is globally generated and the inequalities $2g < \deg(L)$ and $4g - 2h^1(L') < \deg(L) + \deg(L')$ hold, the multiplication map $H^0(L)\otimes H^0(L') \to H^0(L\otimes L')$ is surjective.

Proof. (*of Theorem 3.5*) Together with these remarks we can prove the theorem. First we can assume that $L = \mathcal{O}(C)$ for a smooth curve C (see 3.6). From the second remark 3.7, we see that it is enough to verify surjectivity of the maps

$$H^0(L^r)\otimes H^0(L) \to H^0(L^{r+1})$$

for all $r \geq 2$. By the third remark 3.8, it suffices to prove surjectivity of

$$H^0((L\otimes \mathcal{O}_C)^r)\otimes H^0(L\otimes \mathcal{O}_C) \to H^0((L\otimes \mathcal{O}_C)^{r+1}).$$

Note that for this reduction we have to use the Kawamata-Vieweg vanishing theorem: We need the condition $H^1(L^r\otimes L^{-1}) = H^1(L^{r-1}) = 0$ for all $r \geq 2$, which holds by this theorem. Finally, we want to use Butler's criterion 3.9 and have to check the two inequalities. But these are true as long as $(L.L) \geq 4$.
To see the connection between the multiplication map and the kernel of the evaluation map, we start with the short exact sequence (3.1) for $G = L^r$, tensor it with L^s and look at the induced long exact sequence. This is the same as in the proof of Theorem 3.2:

$$\cdots \to H^0(L^s)\otimes H^0(L^r) \to H^0(L^{r+s}) \to H^1(M_{L^r}\otimes L^s) \to H^0(L^r)\otimes \underbrace{H^1(L^s)}_{=0} \to \cdots .$$

Since $H^1(M_{L^r}\otimes L^s)$ is the cokernel of the multiplication map, it vanishes if and only if the multiplication is surjective. q.e.d.

As a direct consequence, we obtain, together with Theorem 3.2, a result already known by Saint-Donat in 1974. One can find it in [SD74, Theorem 6.1.(ii)]. But the assumptions of Saint-Donat are slightly different form ours.

3.10 Corollary. *Let S be a K3 surface and L a globally generated line bundle on S with $(L.L) \geq 4$. Then L^r is normally generated for $r \geq 2$.*

Now we show the vanishing of the groups that imply the property N_p.

3.11 Theorem. *Let S be a K3 surface. Let L be a globally generated line bundle on S such that $(L.L) \geq 4$. Then*

(i) $H^1(M_{L^r}^{\otimes 2} \otimes L^s) = 0$ *for all* $r, s \geq 2$.

(ii) $H^1(M_{L^r}^{\otimes p+1} \otimes L^s) = 0$ *for all* $p \geq 0, r \geq 2, s \geq p+1$

In this proof we will use the concept of stability. Recall that the slope of a vector bundle, denoted by μ, is its degree[1] divided by its rank. A vector bundle $\mathscr{E}$ is called semistable if the slope of every subbundle is smaller or equal to the slope of $\mathscr{E}$. Note that a line bundle is always semistable.

3.12. In his paper [But94], Butler also gives a criterion for the kernel of the evaluation map to be semistable, Theorem 1.2: If $\mathscr{E}$ is a semistable vector bundle with $\mu(\mathscr{E}) \geq 2g$, then $M_{\mathscr{E}}$ is semistable, too. It also states, that $\mu(M_{\mathscr{E}}) \geq -2$, which will be analyzed in detail in the proof of the theorem. If we look at $\mathscr{E} = L^r$, the tensor power of a line bundle L of degree $2g$, with g the genus of the curve, we see that it is semistable, since all line bundles are semistable, and has slope $\mu(L^r) = \deg(L^r) = 2rg \geq 2g$ for all $r \geq 1$. Hence, M_{L^r} is semistable for all $r \geq 1$.

In the proof we will deal with the bundle $M_{L^r} \otimes L^s \otimes \mathcal{O}_C$, where $C \in |L|$. This bundle has the disadvantage of not being semistable. But there is a way to go from this bundle to a semistable one. A slightly stronger version of the following lemma and its proof can be found in [GP99, Lemma 2.9].

3.13 Lemma. *Let X be a projective variety, let q be a nonnegative integer and let F be a globally generated line bundle on X. Let Q be an effective line bundle on X and C be a reduced and irreducible member of $|Q|$. Let R be a line bundle and G a sheaf on X such that:*

(i) $H^1(F \otimes Q^*) = 0$

(ii) $H^0(M_{F_C}^{\otimes q'} \otimes R_C) \otimes H^0(G) \to H^0(M_{F_C}^{\otimes q'} \otimes R \otimes G \otimes \mathcal{O}_C)$ *is surjective for all* $0 \leq q' \leq q$.

Then the following map is surjective for all $0 \leq p \leq q$ and all $0 \leq k \leq p$:

$$H^0(M_F^{\otimes k} \otimes M_{F_C}^{\otimes p-k} \otimes R_C) \otimes H^0(G) \to H^0(M_F^{\otimes k} \otimes M_{F_C}^{\otimes p-k} \otimes G \otimes R \otimes \mathcal{O}_C).$$

[1] The degree of a vector bundle $\mathscr{E}$ is defined as $\deg(\mathscr{E}) = \deg(\det(\mathscr{E}))$.

Now we have gathered the tools to prove Theorem 3.11.

Proof. We fix some $r, s \geq 2$ and start with sequence (3.1) again, but this time we tensor it with $M_{L^r} \otimes L^s$ and take global sections. The part of the result we have to look at is the following:

$$\begin{tikzcd} 0 \arrow[r] & H^0(M_{L^r}^{\otimes 2} \otimes L^s) \arrow[r] & H^0(L^r) \otimes H^0(M_{L^r} \otimes L^s) \arrow[r, "\alpha"] & H^0(M_{L^r} \otimes L^{r+s}) \arrow[dll] \\ & H^1(M_{L^r}^{\otimes 2} \otimes L^s) \arrow[r] & H^1(M_{L^r} \otimes L^s) \arrow[r] & \cdots . \end{tikzcd}$$

The last term is zero, due to the vanishing of $H^1(M_{L^r} \otimes L^s)$, which was the statement of Theorem 3.5. Now our aim is to show the surjectivity of the map α. As in the last proof, we reduce via 3.7 to requiring surjectivity of

$$H^0(L) \otimes H^0(M_{L^r} \otimes L^s) \to H^0(M_{L^r} \otimes L^{s+1})$$

for all $r, s \geq 2$. The remark 3.8 allows us to show that the map

$$H^0(L \otimes \mathcal{O}_C) \otimes H^0(M_{L^r} \otimes L^s \otimes \mathcal{O}_C) \to H^0(M_{L^r} \otimes L^{s+1} \otimes \mathcal{O}_C)$$

is onto. The curve C in the system $|L|$ is chosen via 3.6. As usual, we abbreviate $L \otimes \mathcal{O}_C$ with L_C. Thus, we can write $H^0(L_C) \otimes H^0(M_{L^r} \otimes L_C^s) \to H^0(M_{L^r} \otimes L_C^{s+1})$.
We now want to pass to a semistable bundle and continue working with it. To do so, we will apply Lemma 3.13. We choose $q = 1$ and $F = L^r$, which is globally generated – simply because L is. The effective line bundle Q is taken to be $L = \mathcal{O}(C)$, $R = L^s$ and $G = L_C$. Now the lemma, for $k = p = 1$, implies the surjection we are interested in. To establish the conditions of the lemma, it is sufficient to show that $H^1(F \otimes Q^*) = H^1(L^{r-1}) = 0$ and

$$H^0(M_{L_C^r}^{\otimes p} \otimes L_C^s) \otimes H^0(L_C) \to H^0(M_{L_C^r}^{\otimes p} \otimes L^s \otimes \mathcal{L}_C \otimes \mathcal{O}_C) = H^0(M_{L_C^r}^{\otimes p} \otimes L_C^s)$$

is surjective for $p = 0, 1$. The vanishing of $H^1(L^{r-1})$ for $r \geq 1$ has already been seen and used. The second condition actually consists of two conditions. The first one, for $k = 0$, translates as surjectivity of the map $H^0(L_C^s) \otimes H^0(L_C^s) \to H^0(L_C^{s+1})$, which was shown in the last proof using Butler's criterion 3.9. In the case $k = 1$, we have to verify that the map

$$H^0(L_C) \otimes H^0(M_{L_C^r} \otimes L_C^s) \to H^0(M_{L_C^r} \otimes L_C^{s+1})$$

is surjective for all $r, s \geq 2$. The cokernel of this map is $H^1(M_{L_C^r} \otimes M_{L_C} \otimes L_C^s)$. As we have seen in remark 3.12, the bundle $M_{L_C^t}$ is semistable for all $t \geq 1$. Now we use the tensor product theorem, stating that tensor products of semistable bundles are semistable, see for example [Miy85, Corollary 3.7], and deduce the semistability of the bundle $E := M_{L_C^r} \otimes M_{L_C} \otimes L_C^s$. We now estimate the slope of E, starting with $M_{L_C^r}$. Let g be the genus of the curve C. Note that, due to $4 \leq (L.L) = 2g - 2$, the genus is at least 3. We then have $\mathrm{rk}(M_{L_C^r}) = h^0(L_C^r) - 1 = \deg(L_C^r) - g = 2rg - g$ and $\deg(M_{L_C^r}) = -\deg(L_C^r) = -2rg$ for all r. The case $r = 1$, which is also a term in E, must therefore have slope $\mu(M_{L_C}) = -2$. If $r \geq 2$ we can estimate the slope via $r > 1$:

$$\mu(M_{L_C^r}) = \frac{\deg M_{L_C^r})}{\mathrm{rk}(M_{L_C^r})} = \frac{-2rg}{2rg - g} > \frac{-2g}{2g - g} = -2.$$

Composing these we obtain:

$$\begin{aligned}\mu(E) &= \mu(M_{L_C^r} \otimes M_{L_C} \otimes L_C^s) \\ &= \mu(M_{L_C^r}) + \mu(M_{L_C}) + \mu(L_C^s) > -2 - 2 + 2gs > 2g - 2 = \mu(\omega_C),\end{aligned}$$

where ω_C denotes the canonical bundle on C. We can rewrite this as

$$\mu(E^* \otimes \omega_C) = -\mu(E) + \mu(\omega_C) < (-2g + 2) + (2g - 2) = 0.$$

The vector bundle E is semistable, so is ω_C, and hence, $E^* \otimes \omega_C$ is also semistable. But a semistable bundle with a negative slope has no global sections: $H^0(E^* \otimes \omega_C) = 0$. Finally, Serre Duality yields $H^0(E^* \otimes \omega_C) = H^1(E)^*$ and gives us the vanishing we wanted.

The second part of the theorem is done essentially in the same way. The result was already proven for $p = 0$ in Theorem 3.5 and for $p = 1$ in the first part of this proof – we use this as the basis for an induction on p.

Thus, let us assume the statement holds for p. As in the first part, one starts with the sequence (3.1) and twists it with $M_{L^r}^{\otimes p} \otimes L^s$. By induction, the term $H^1(M_{L^r}^{\otimes p} \otimes L^s)$ vanishes for all $r \geq 2$ and $s \geq p$, therefore we have to show the surjectivity of the map corresponding to α; recall the beginning of the proof of the first part. By the remark 3.7 and 3.8 the problem is reduced to the surjectivity of

$$H^0(M_{L^r}^{\otimes p} \otimes L^s \otimes \mathcal{O}_C) \otimes H^0(L \otimes \mathcal{O}_C) \to H^0(M_{L^r}^{\otimes p} \otimes L^{s+1} \otimes \mathcal{O}_C).$$

Now we use Lemma 3.13 again with the same input: The bundle G is chosen as L_C, $Q = L$ is still an effective line bundle with C as a general member, $F = L^r$ is globally generated and $R = L^s$. We have only changed the integer q from 1 to p. As seen before, the lemma yields the right surjection if the conditions are satisfied. Since we have only changed the parameter q, the first condition is still satisfied for the same reason. The second one actually consists of more than two conditions, since p' now ranges from 0 to p. The cases $p' = 0$ and $p' = 1$ have already been discussed, so we assume $p' > 1$. The map whose surjection we are interested in is

$$H^0(M_{L_C^r}^{\otimes p'} \otimes L_c^s) \otimes H^0(L_C) \to H^0(M_{L_C^r}^{\otimes p'} \otimes L_C^{s+1})$$

and has the cokernel $H^1(M_{L_C^r}^{\otimes p'} \otimes M_{L_C} \otimes L_C^s)$. As we have seen before, the bundle $E = M_{L_C^r} \otimes M_{L_C} \otimes L_C^s$ is semistable for all r, s. Hence, the bundle

$$E_{p'} = E \otimes M_{L_C^r}^{\otimes p'-1} = M_{L_C^r}^{\otimes p'} \otimes M_{L_C} \otimes L_C^s$$

is also semistable due to the tensor product theorem. Using the same arguments as before, we can deduce the vanishing of $H^0(E_{p'} \otimes \omega_C) = H^1(E_{p'})^*$ from our estimate of the slope. This yields the desired surjectivity. q.e.d.

As a direct consequence of 3.11 together with Theorems 3.2 and 3.4 we obtain:

3.14 Corollary. *Let S be a K3 surface and L a globally generated line bundle on S with $(L.L) \geq 4$. Then L^r is normally generated for $r \geq 2$. Moreover, if $s \geq p+1$, then L^s satisfies N_p.*

These results state the vanishing of the first cohomology groups. Later on, when we study the property N_p on the product, Theorem 3.4 will be applied. The kernel of the evalution map involved will be expressed as a sum of certain vector bundles. To deal with the cohomology of such an object, we proved Lemma 2.2. For this lemma, we require the vanishing of higher cohomology groups, which is precisely covered by the following

3.15 Lemma. *Let $\mu \in \mathbb{N}$, X be a projective variety and L a globally generated line bundle such that $H^i(L^t) = 0$ for all $i \geq 1$ and all $t \geq \mu$. Then,*

(i) $H^i(M_{L^r} \otimes L^s) = 0$ *for all* $i \geq 2, r \geq 1$ *and* $s \geq \mu$

(ii) $H^i(M_{L^r}^{\otimes p+1} \otimes L^s) = 0$ *for all* $p \geq 0, i \geq 3, r \geq 1$ *and* $s \geq \mu$

(iii) If in addition $H^1(M_{L^r}^{\otimes p'+1} \otimes L^s)$ *vanishes for all* $0 \leq p' \leq p, r \geq 1$ *and* $s \geq \mu$*, then* $H^2(M_{L^r}^{\otimes p'+1} \otimes L^s) = 0$ *for all* $0 \leq p' \leq p, r \geq 1$ *and* $s \geq \mu$*. Here it suffices to ask for the vanishing of* $H^1(L^t)$ *and* $H^2(L^t)$ *for all* $t \geq \mu$.

Proof. The vanishing of $H^i(M_{L^r} \otimes L^s)$ for $i \geq 2$ can be deduced from the sequence (3.1) for $G = L^r$ by tensoring it with L^s and taking global sections:

$$\cdots \to \underbrace{H^{i-1}(L^{r+s})}_{=0\ \forall r+s \geq \mu} \to H^i(M_{L^r} \otimes L^s) \to H^0(L^r) \otimes \underbrace{H^i(L^s)}_{=0\ \forall s \geq \mu} \to \cdots$$

So $H^i(M_{L^r} \otimes L^s)$ also vanishes for all $i \geq 2$, $r \geq 2$ and $s \geq \mu$.
For the second part, we induct on p. The case $p = 0$ is the first statement, so we assume the vanishing for p. Again, we start with sequence (3.1). This time we tensor with $M_{L^r}^{\otimes p} \otimes L^s$ and take global sections. We obtain

$$\cdots \to \underbrace{H^{i-1}(M_{L^r}^{\otimes p} \otimes L^{r+s})}_{=0\ \forall r \geq 1, r+s \geq \mu} \to H^i(M_{L^r}^{\otimes p+1} \otimes L^s) \to H^0(L^r) \otimes \underbrace{H^i(M_{L^r}^{\otimes p} \otimes L^s)}_{=0\ \forall r \geq 1, s \geq \mu} \to \cdots$$

Clearly, the middle term of this sequence vanishes for $i \geq 3$,$r \geq 1$ and $s \geq \mu$.
For the last part, we again induct on p and look at the beginning of the same long exact sequence:

$$\cdots \to \underbrace{H^1(M_{L^r}^{\otimes p} \otimes L^{r+s})}_{\substack{=0\ \forall r \geq 1, r+s \geq \mu \\ \text{by assumtion}}} \to H^2(M_{L^r}^{\otimes p+1} \otimes L^s) \to H^0(L^r) \otimes \underbrace{H^2(M_{L^r}^{\otimes p} \otimes L^s)}_{\substack{=0\ \forall r \geq 1, s \geq \mu \\ \text{by induction}}} \to \cdots$$

q.e.d.

Let L be a globally generated line bundle on a K3 surface such that $(L.L) \geq 4$. Then we have:

$$H^i(M_{L^r}^{\otimes p+1} \otimes L^s) = 0 \qquad \forall p \geq 0, i \geq 1, r \geq 2, s \geq p+1 \tag{3.16}$$

By the Kawamata-Viehweg vanishing theorem, one has $H^i(L^t) = 0$ for all $i \geq 1$ and for all $t \geq 1 = \mu$. So we can apply the second part of Lemma 3.15 and get the vanishing for all $i \geq 3$, $p \geq 0$ and $r, s \geq 1$. By Theorems 3.5 and 3.11 we obtain the vanishing for $i = 1$, $p = 0$, $r \geq 1$ and $s \geq p+1$. Now, we can apply the third part of the previous lemma and conclude the remaining vanishing statements for $i = 2$.

q.e.d.

As mentioned in the beginning, we will show the normal generation of special line bundles on the symmetric product and the Hilbert scheme of a K3 surface. Of course, we will use Theorem 3.2 and show the vanishing of the cohomology groups in question. The situation can be roughly described as follows: We have a line bundle K on the 2-nd symmetric power and L the corresponding line bundle on the 2-fold tensor power. The vanishing of $H^1(M_{L^r} \otimes L^s)$ is known for some r and s and we have to relate it to the vanishing of $H^1(M_{K^r} \otimes K^s)$, where the following lemma leads us to.

3.17 Lemma. *Let S be a K3 surface and L a line bundle on S such that $(L.L) \geq 4$. We obtain a line bundle $J := L \boxtimes L$ on the product $S \times S$. Since this bundle is invariant under the operation of $\mathfrak{S}_2$, it descends to a bundle K on the symmetric product $S^2(S)$ and therefore, it has the property $p^*K = J$. As in the diagram (2.7) $p : S^2 \to S^2(S)$ is the quotient map.*
Now the following holds: $H^1(p^*M_{K^r} \otimes J^s) = 0$ for all $r, s \geq 2$.

Proof. We start with the usual sequence (3.1), pull it back via p^* and take global sections. As a result, we get

$$\begin{tikzcd} 0 \arrow[r] & H^0(p^*M_{K^r} \otimes J^s) \arrow[r] & p^*H^0(K^r) \otimes H^0(J^s) \arrow[r] & H^0(J^{r+s}) \arrow[dll] \\ & H^1(p^*M_{K^r} \otimes J^s) \arrow[r] & p^*H^0(K^r) \otimes H^1(J^s) = 0 \arrow[r] & \cdots \end{tikzcd} \tag{3.18}$$

Since the pullback of K is the $\mathfrak{S}_2$-invariant line bundle J, we see that $p^*H^0(K^r)$ is the part of $H^0(J^r) = H^0((L \boxtimes L)^r) = H^0(L^r) \otimes H^0(L^r)$ which is invariant under the operation of the symmetric group $\mathfrak{S}_2$, which exchanges the components. Hence, it is isomorphic to $S^2(H^0(L^r))$. Looking at the sequence (3.18) and filling in these isomorphisms, we see that our task is to show the surjectivity of the map

$$m_s : S^2(H^0(L^r)) \otimes H^0(L^s)^{\otimes 2} \to H^0(L^{r+s}) \otimes H^0(L^{r+s}),$$

where we multiply the first and third as well as the second and fourth components, respectively. This multiplication $m : H^0(L^r) \otimes H^0(L^s) \to H^0(L^{r+s})$ is surjective due to Theorem 3.5. Since the tensor product is commutative, we see that $m(x \otimes y) = m(y \otimes x)$

for all $x \in H^0(L^r)$ and $y \in H^0(L^s)$. For ease of notation, we will abbreviate $m(x \otimes y)$ by xy.
Next, we show that the map m_s is onto by lifting $a \otimes b \in H^0(L^{r+s})^{\otimes 2}$ as an arbitrary generator. Since m is surjective, we can pick preimages of a and b. Furthermore, there are two cases: $r \geq s$ and $s \geq r$. Since we will use this lemma to prove property N_p, we restrict ourself to the case $s \geq r$ and write $s = r + t$. For sake of simplicity we assume that a and b lift to pure tensors. Thus, we obtain $a = m(x_a \otimes y_a) = x_a y_a$ and $b = m(x_b \otimes y_b) = x_b y_b$ for some $x_a, x_b \in H^0(L^r)$ and some $y_a, y_b \in H^0(L^s)$. We now have another multiplication map $H^0(L^r) \otimes H^0(L^t) \to H^0(L^s)$ allowing us to lift the elements y_a and y_b. We write $y_a = y_a^r y_a^t$ and $y_b = y_b^r y_b^t$ using this multiplication map. We claim that the following element is a preimage of $a \otimes b$:

$$
\begin{aligned}
&(x_a \otimes x_b + x_b \otimes x_a) \otimes y_a \otimes y_b + \frac{1}{2}(x_a \otimes y_b^r + y_b^r \otimes x_a) \otimes x_b y_a^t \otimes y_a^r y_b^t \\
-&\frac{1}{2}(x_a \otimes y_a^r + y_a^r \otimes x_a) \otimes x_b y_a^t \otimes y_b - \frac{1}{2}(y_b^r \otimes y_a^r + y_a^r \otimes y_b^r) \otimes x_b y_a^t \otimes x_a y_b^t
\end{aligned}
$$

.

By calculation, we will establish that this element actually is mapped to $a \otimes b$, using $xy = m(x \otimes y) = m(y \otimes x)$ for suitable x and y.

$$
\begin{aligned}
m_s\Big(&(x_a \otimes x_b + x_b \otimes x_a) \otimes y_a \otimes y_b + \frac{1}{2}(x_a \otimes y_b^r + y_b^r \otimes x_a) \otimes x_b y_a^t \otimes y_a^r y_b^t \\
&- \frac{1}{2}(x_a \otimes y_a^r + y_a^r \otimes x_a) \otimes x_b y_a^t \otimes y_b - \frac{1}{2}(y_b^r \otimes y_a^r + y_a^r \otimes y_b^r) \otimes x_b y_a^t \otimes x_a y_b^t\Big) \\
=& x_a y_a \otimes x_b y_b \underbrace{+ x_b y_a \otimes x_a y_b}_{①} \underbrace{+ \frac{1}{2} x_a x_b y_a^t \otimes y_b^r y_a^r y_b^t}_{②} \underbrace{+ \frac{1}{2} y_b^r x_b y_a^t \otimes x_a y_a^r y_b^t}_{③} \\
& \underbrace{- \frac{1}{2} x_a x_b y_a^t \otimes y_a^r y_b}_{②} \underbrace{- \frac{1}{2} y_a^r x_b y_a^t \otimes x_a y_b}_{①} \underbrace{- \frac{1}{2} y_b^r x_b y_a^t \otimes y_a^r x_a y_b^t}_{③} \underbrace{- \frac{1}{2} y_a^r x_b y_b^t \otimes y_b^r x_a y_b^t}_{①} \\
=& x_a y_a \otimes x_b y_b = a \otimes b.
\end{aligned}
$$

The terms with the same number cancel each other, when using $y_a^t y_a^s = y_a$ and $y_b^t y_b^s = y_b$.
q.e.d.

We get a similar lemma for the Hilbert scheme.

3.19 Lemma. *Let S be a K3 surface and L a line bundle on S, generated by global sections with $(L.L) \geq 4$. Then we pull back the bundle $L \boxtimes L$ to the blowup along the diagonal and denote this bundle with $\mathcal{L}$. This bundle descends to a bundle $\mathcal{J}$ on the Hilbert scheme $\mathrm{Hilb}^2(S)$ and therefore has the property $\widetilde{p}^* \mathcal{J} = \mathcal{L}$. The map $\widetilde{p} : \widetilde{S \times S} \to \mathrm{Hilb}^2(S)$ is the quotient by the $\mathfrak{S}_2$-operation (cf. (2.7)). The bundles satisfy $H^1(\widetilde{p}^* M_{\mathcal{J}^r} \otimes \mathcal{L}^s) = 0$ for all $r, s \geq 2$.*

Proof. We reduce the statement to the previous lemma by "lifting it along a blowup". If we look at diagram (2.7) we have proven the statement for the right map p and want to show it for the left map $\widetilde{p}$. We will use the notation of this diagram.

$$\begin{aligned}
0 &= H^1(p^*M_{K^r} \otimes J^s) \\
&= H^1(p^*M_{K^r} \otimes p^*K^s \otimes \varphi_*\mathcal{O}_{S^2(S)}) && \text{Theorem 2.4} \\
&= H^1(\varphi_*(\varphi^*(p^*(M_{K^r} \otimes K^s))) \otimes \mathcal{O}_{S^2(S)}) && \text{Leray spectral sequence} \\
&= H^1((p \circ \varphi)^*(M_{K^r} \otimes K^s)) \\
&= H^1((\bar{\varphi} \circ \widetilde{p})^*(M_{K^r} \otimes K^s)) \\
&= H^1(\widetilde{p}^* M_{\bar{\varphi}^*K^r} \otimes \bar{\varphi}^*K^s) && \text{Theorem 2.4} \\
&= H^1(\widetilde{p}^* M_{\mathcal{J}^r} \otimes \mathcal{L}^s).
\end{aligned}$$

q.e.d.

Properties of M_G

As we have seen, the normal generation of a line bundle can be checked by calculating the cohomology group $H^1(M_L \otimes L^k)$ for all k. Hence, we need some more techniques to handle M_L. The following theorem plays a crucial role in this thesis. It allows us to lift property N_p to the product of varieties. More precisely, if we have two varieties with a line bundle on each of them, it provides a formula for $M_{L\boxtimes L'}$ on the product.

3.20 Theorem. *Let X and X' be two projective varieties, L a globally generated line bundle on X and L' a globally generated line bundle on X'. Then for the line bundle $J := \pi_1^*L \otimes \pi_2^*L'$ on $Y = X \times X'$, the following formula holds:*

$$M_J = \pi_1^*M_L \otimes \pi_2^*(D') + \pi_1^*(D) \otimes \pi_2^*M_{L'} \ \subseteq \pi_1^*D \otimes \pi_2^*D'$$

with $D = H^0(L) \otimes \mathcal{O}_X$ and $D' = H^0(L') \otimes \mathcal{O}_{X'}$.

Proof. Since M_J is the kernel of the map (3.1), we start by analyzing $H^0(J) \otimes \mathcal{O}_Y$. We denote by ρ_Y (resp. ρ_X) the map from Y (resp. X) to the point $\mathrm{Spec}(\mathbb{C})$. Furthermore, we know that $H^0(J) \otimes \mathcal{O}_Y = \rho_Y^*\rho_{Y,*}J$ and the evaluation map is the counit.
So we want to calculate $\rho_Y^*\rho_{Y,*}J$, starting with

$$\rho_{Y,*}J = \rho_{Y,*}(\pi_1^*L \otimes \pi_2^*L') = H^0(X \times X', \pi_1^*L \otimes \pi_2^*L') = H^0(X, L) \otimes H^0(X', L')$$

In the next step we compute $\rho_Y^*\rho_{Y,*}J$:

$$\begin{aligned}
\rho_Y^*\rho_{Y,*}J &= \rho_Y^*(H^0(L) \otimes H^0(L')) \\
&= \rho_Y^*H^0(L) \otimes \rho_Y^*H^0(L') \\
&= \pi_1^*(\rho_X^*H^0(L)) \otimes \pi_2^*(\rho_{X'}^*H^0(L')) \\
&= \pi_1^*(H^0(L) \otimes \mathcal{O}_X) \otimes \pi_2^*(H^0(L') \otimes \mathcal{O}_X') \\
&= \pi_1^*(D) \otimes \pi_2^*(D').
\end{aligned}$$

Finally, with the help of remark (2.3), this gives the equation for M_J:

$$\begin{aligned} M_J &= \ker(\rho_Y^*\rho_{Y,*}J \to J) \\ &= \ker(\pi_1^*(D) \otimes \pi_2^*(D') \to \pi_1^* L \otimes \pi_2^* L') \\ &= \ker(\pi_1^*(e_L) \otimes \pi_2^*(e_{L'})) \\ &= \ker(\pi_1^*(e_L)) \otimes \pi_2^*(D') + \pi_1^*(D) \otimes \ker(\pi_2^*(e_{L'})) \\ &= \pi_1^*(M_L) \otimes \pi_2^*(D') + \pi_1^*(D) \otimes \pi_2^*(M'_L) \qquad\qquad \subseteq \pi_1^*(D) \otimes \pi_2^*(D'). \end{aligned}$$

q.e.d.

3.21 Theorem. *Let L be a globally generated line bundle on a projective variety X and $\varphi : \widetilde{X} \to X$ a blowup along some subvariety. Then $\varphi^* M_L = M_{\varphi^* L}$.*

Proof. We use the notation and the idea of the previous proof and start by analyze $H^0(\varphi^* L) \otimes \mathcal{O}_{\widetilde{X}}$. The projection formula gives us:

$$\rho_{\widetilde{X}}^* \rho_{\widetilde{X},*} \varphi^*(L) = \varphi^* \rho_X^* \rho_{X,*} \varphi_*(\varphi^* L \otimes \mathcal{O}_{\widetilde{X}}) = \varphi^* \rho_X^* \rho_{X,*}(L \otimes \varphi_* \mathcal{O}_{\widetilde{X}}) = \varphi^*(H^0(L) \otimes \mathcal{O}_X).$$

In the last equality we use Theorem 2.4: $\varphi_* \mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$.
Since $\varphi^*(M_L)$ is the kernel of the natural map $\varphi^*(H^0(L) \otimes \mathcal{O}_X) \to \varphi^* L$, this implies $\varphi^* M_L = M_{\varphi^* L}$.

q.e.d.

Chapter 4

Normal generation of line bundles

In this chapter we will prove the main theorems of this thesis: In the first section we show that a line bundle $(L \boxtimes L)^r$ on the self-product of a K3 surface is normally generated for every $r \geq 2$ and has property N_p for all $r \geq p+1$. This will be achieved by calculating the cohomology group $H^1(M_{(L\boxtimes L)^r} \otimes (L \boxtimes L)^s)$. For this purpose, we have developed a formula to express $M_{(L\boxtimes L)^r}$ in terms of M_{L^r} in the last chapter.
In the second section we show that normal generation and property N_p are compatible with blowups: If a line bundle has one of these properties, the bundle obtained by pullback also does.
Finally, we look at bundles on the Hilbert scheme and the symmetric power. If we have a bundle on the product (or its blowup) which is normally generated, the bundle obtained by descent is also normally generated.

Line bundles on $S \times S$

4.1 Theorem. *Let X and X' be projective varieties. Let L be a globally generated line bundle on X such that*

- $H^1(M_{L^r} \otimes L^s) = 0$ *for all* $r \geq 1, s \geq \mu$ *(for some* $\mu \in \mathbb{N}$*)*
- $H^i(L^t) = 0$ *for all* $i \geq 1,\ t \geq \mu$

Let L' a globally generated line bundle on X' with the same properties.
Consider the line bundle $J = L \boxtimes L'$ on $X \times X'$. Then

$$H^1(M_{J^r} \otimes J^s) = 0 \qquad \forall r \geq 1,\ s \geq \mu$$

Proof. First we use Theorem 3.20 to express M_J in terms of M_L and $M_{L'}$:

$$\begin{aligned} M_{J^r} \otimes J^s &= \pi_1^* M_{L^r} \otimes \pi_2^*(H^0(L'^r) \otimes \mathcal{O}_{X'}) \otimes \pi_1^* L^s \otimes \pi_2^* L'^s \\ &\quad + \pi_1^*(H^0(L^r) \otimes \mathcal{O}_X) \otimes \pi_2^* M_{L'^r} \otimes \pi_1^* L^s \otimes \pi_2^* L'^s \\ &= \underbrace{\pi_1^*(M_{L^r} \otimes L^s) \otimes \pi_2^*(H^0(L'^r) \otimes L'^s)}_{=:N_1} + \underbrace{\pi_1^*(H^0(L^r) \otimes L^s) \otimes \pi_2^*(M_{L'^r} \otimes L'^s)}_{=:N_2} \\ &= N_1 + N_2 \end{aligned}$$

Lemma 2.2 tells us that we can check the vanishing of $H^1(M_{J^r} \otimes J^s) = H^1(N_1 + N_2)$ for all $r \geq 1$ and $s \geq \mu$ by checking the vanishing of $H^1(N_1)$, $H^1(N_2)$ and $H^2(N_1 \cap N_2)$.

(i) $H^1(N_1) = 0$:
With the help of the Künneth formula and the assumptions, we obtain:

$$\begin{aligned} H^1(N_1) &= H^1\Big(\pi_1^*(M_{L^r} \otimes L^s) \otimes \pi_2^*(H^0(L'^r) \otimes L'^s)\Big) \\ &= \Big(\underbrace{H^1(M_{L^r} \otimes L^s)}_{=0} \otimes H^0(L'^r) \otimes H^0(L'^s)\Big) \\ &\qquad \oplus \Big(H^0(M_{L^r} \otimes L^s) \otimes H^0(L'^r) \otimes \underbrace{H^1(L'^s)}_{=0}\Big) \\ &= 0. \end{aligned}$$

(ii) $H^1(N_2) = 0$:
This can be done analogously to the first part.

(iii) $H^2(N_1 \cap N_2) = 0$
Since $H^i(L^t) = 0$ for all $i \geq 1, t \geq \mu$, we can apply the first part of Lemma 3.15 and obtain $H^2(M_{L^r} \otimes L^s) = 0$ for all $r \geq 1, s \geq \mu$. The same is, of course, true for L'. So the Künneth formula yields:

$$\begin{aligned} H^2(N_1 \cap N_2) &= H^2(\pi_1^*(M_{L^r} \otimes L^s) \otimes \pi_2^*(M_{L'^r} \otimes L'^s)) \\ &= H^0(M_{L^r} \otimes L^s) \otimes \underbrace{H^2(M_{L'^r} \otimes L'^s)}_{=0} \\ &\qquad \oplus \underbrace{H^1(M_{L^r} \otimes L^s)}_{=0} \otimes \underbrace{H^1(M_{L'^r} \otimes L'^s)}_{=0} \\ &\qquad \oplus \underbrace{H^2(M_{L^r} \otimes L^s)}_{=0} \otimes H^0(M_{L'^r} \otimes L'^s) \\ &= 0. \end{aligned}$$

q.e.d.

4.2. There exists another proof of this theorem which is more concrete. But the above proof has the advantage, that it can be seen as a special case of Theorem 4.4.
Starting with the sequence (3.1) for $G = J^r$, twist it with J^s and take global sections to obtain

$$\cdots \to H^0(J^r) \otimes H^0(J^s) \to H^0(J^{r+s}) \xrightarrow{\alpha} H^1(M_{J^r} \otimes J^s) \to H^0(J^r) \otimes \underbrace{H^1(J^s)}_{=0} \to \cdots .$$

The last term is zero due to

$$H^1(J^r) = H^1((L \boxtimes L')^r) = H^0(L^r) \otimes \underbrace{H^1(L'^r)}_{=0} \oplus \underbrace{H^1(L^r)}_{=0} \otimes H^0(L'^r) = 0$$

for all r. We conclude that $H^1(M_{J^r} \otimes J^s) = 0$ is equivalent to the surjectivity of α. Using the Künneth formula, α can be written as a tensor product of two other multiplications:

$$\begin{tikzcd} H^0(J^r) \otimes H^0(J^s) \arrow[r, "\alpha"] \arrow[d, equal] & H^0(J^{r+s}) \arrow[d, equal] \\ H^0(L^r) \otimes H^0(L'^r) \otimes H^0(L^s) \otimes H^0(L'^s) \arrow[r] & H^0(L^{r+s}) \otimes H^0(L'^{r+s}). \end{tikzcd}$$

The bottom map is surjective due to Theorem 3.5. Hence, α is also surjective for all $r \geq 1$ and $s \geq \mu$.

As a corollary, we recover the analogue to Theorem 3.5 for the self-product of a K3 surfaces.

4.3 Corollary. *Let S be a K3 surface and L a line bundle on S with $(L.L) \geq 4$, generated by its global sections. Let J be the line bundle $L \boxtimes L$ on $S \times S$. Then J^r is normally generated for all $r \geq 2$.*

Proof. By Theorem 3.2 it suffices to prove $H^1(M_{J^r} \otimes J^s) = 0$ for all $r, s \geq 2$ to get the normal generation.
For this vanishing we want to use the above Theorem 4.1. In order to apply it with $X = X' = S$, $\mu = 2$ and $L = L'$, we have to verify two conditions:

- $H^1(M_{L^r} \otimes L^s) = 0 \qquad \forall r \geq 1, s \geq 2$
  This is true by Theorem 3.5, since S is a K3 surface.
- $H^i(L^t) = 0 \qquad \forall i \geq 1 t \geq 2$
  This is true by the Kawamata-Viehweg vanishing theorem.

Thus, we can apply Theorem 4.1 and deduce the vanishing required for Theorem 3.2. q.e.d.

We get the results required for property N_p.

4.4 Theorem. *Let S be a K3 surface and L a globally generated line bundle, such that $(L.L) \geq 4$. Again, we define $X := S \times S$ and $J := \pi_1^* L \otimes \pi_2^* L$. Then,*

$$H^1(M_{J^r}^{\otimes p+1} \otimes J^s) = 0 \textit{ for all } r \geq 2, s \geq p+1.$$

Proof. In order to calculate this cohomology group we introduce some notation:
$\Xi = \{I_1 \otimes \ldots \otimes I_{p+1} \subseteq (H^0(L^r) \otimes \mathcal{O}_S)^{\otimes p+1} | \forall i : I_i \in \{M_{L^r}, H^0(L^r) \otimes \mathcal{O}_S\}\}$ and for any $I = I_1 \otimes \cdots \otimes I_{p+1} \in \Xi$ the "dual" $\widehat{I} = \widehat{I_1} \otimes \ldots \otimes \widehat{I_{p+1}} \in \Xi$ via:

$$\widehat{I_i} = \begin{cases} M_{L^r} & \text{if } I_i = H^0(L^r) \otimes \mathcal{O}_S \\ H^0(L^r) \otimes \mathcal{O}_S & \text{if } I_i = M_{L^r} \end{cases}$$

With this terminology we start calculating the cohomology group:

$$\begin{aligned}
&H^1(M_{J^r}^{\otimes p+1} \otimes J^s)\\
&\quad = H^1\Big([\pi_1^* M_{L^r} \otimes \pi_2^*(H^0(L^r) \otimes \mathcal{O}_S) + \pi_1^*(H^0(L^r) \otimes \mathcal{O}_S) \otimes \pi_2^* M_{L^r}]^{\otimes p+1}\\
&\qquad\quad \otimes \pi_1^* L^s \otimes \pi_2^* L^s\Big)\\
&\quad = H^1\Big(\sum_{I \in \Xi} (\pi_1^* I \otimes \pi_2^* \widehat{I}) \otimes \pi_1^* L^s \otimes \pi_2^* L^s\Big)\\
&\quad = H^1\Big(\sum_{I \in \Xi} \pi_1^*(I \otimes L^s) \otimes \pi_2^*(\widehat{I} \otimes L^s)\Big)
\end{aligned}$$

We want to use Lemma 2.2 to make sure that this group vanishes. Thus, we have to show the vanishing of the cohomology groups $H^{|\Lambda|}(\bigcap_{I\in\Lambda} \pi_1^*(I \otimes L^s) \otimes \pi_2^*(\widehat{I} \otimes L^s))$ for all non-empty $\Lambda \subseteq \Xi$. Since $M_{L^r} \subseteq H^0(L^r) \otimes \mathcal{O}_S$, we obtain for the intersection that $\bigcap_{I\in\Lambda} \pi_1^*(I \otimes L^s) \otimes \pi_2^*(\widehat{I} \otimes L^s) = \pi_1^*(A_1 \otimes \ldots \otimes A_{p+1} \otimes L^s) \otimes \pi_2^*(B_1 \otimes \ldots \otimes B_{p+1} \otimes L^s)$ with

$$A_i = \begin{cases} M_{L^r} & \text{if } \exists I_1 \otimes \ldots \otimes I_{p+1} \in \Lambda : I_i = M_{L^r} \\ H^0(L^r) \otimes \mathcal{O}_S & \text{if } \forall I_1 \otimes \ldots \otimes I_{p+1} \in \Lambda : I_i = H^0(L^r) \otimes \mathcal{O}_S \end{cases}$$

$$B_i = \begin{cases} M_{L^r} & \text{if } \exists I_1 \otimes \ldots \otimes I_{p+1} \in \Lambda : I_i = H^0(L^r) \otimes \mathcal{O}_S \\ H^0(L^r) \otimes \mathcal{O}_S & \text{if } \forall I_1 \otimes \ldots \otimes I_{p+1} \in \Lambda : I_i = M_{L^r}. \end{cases}$$

By the Künneth formula:

$$H^{|\Lambda|}(\pi_1^*(A \otimes L^s) \otimes \pi_2^*(B \otimes L^s)) = \bigoplus_{a+b=|\Lambda|} H^a(A \otimes L^s) \otimes H^b(B \otimes L^s) \tag{4.5}$$

where $A = A_1 \otimes \ldots \otimes A_{p+1}$ and $B = B_1 \otimes \ldots \otimes B_{p+1}$.
As we do not want to intersect them, we are allowed to permute the factors. This gives A (resp. B) a well-arranged form: $A = M_{L^r}^{\otimes\mu} \otimes H^0(L^r)^{\otimes\nu}$ for some $\mu, \nu \geq 0$ with $\mu + \nu = p+1$ (resp. $B = M_{L^r}^{\otimes\mu'} \otimes H^0(L^r)^{\otimes\nu'}$)
Hence, the cohomology group we are interested in can be written as

$$H^a(A \otimes L^s) = H^a(M_{L^r}^{\otimes\mu} \otimes H^0(L^r)^{\otimes\nu} \otimes L^s) = H^0(L^r)^{\otimes\nu} \otimes H^a(M_{L^r}^{\otimes\mu} \otimes L^s).$$

We distinguish three different cases:

(i) $\mu, \nu \geq 1$:

a) $a \geq 1$:
This is the general case: $H^a(M_{L^r}^{\otimes\mu} \otimes L^s)$ with $a, \mu \geq 1, r \geq 2$ and $s \geq p+1$ is vanishing because of remark (3.16).

b) $a = 0$:
In this case we get $b = |\Lambda| \geq 1$ and look at the other factor in (4.5): $H^b(B \otimes L^s)$. Since $\nu \geq 1$ we know that there exits an $j \in \{1, \dots, p+1\}$ such that for all $I_1 \otimes \dots \otimes I_{p+1} \in \Lambda$ we have $I_j = H^0(L^r) \otimes \mathcal{O}_S$. Hence, B can be written as $M_{L^r}^{\otimes \mu'} \otimes H^0(L^r)^{\otimes \nu'}$ with $\mu' \geq 1$. Thus we have again the general case: $H^b(B \otimes L^s) = H^b(M_{L^r}^{\otimes \mu'} \otimes L^s) \otimes H^0(L^r)^{\otimes \nu'}$ with $b, \mu' \geq 1, r \geq 2$ and $s \geq p+1$

(ii) $\mu = 0$:
We look at the definition of μ: The vanishing of μ means that for every $j = 1, \dots, p+1$ and for every $I_1 \otimes \dots \otimes I_{p+1} \in \Lambda$: $I_j = H^0(L^r \otimes \mathcal{O}_S)$. Hence, Λ consists of only one element, namely $(H^0(L^r) \otimes \mathcal{O}_S)^{\otimes p+1}$. Consider the intersection:

$$
\begin{aligned}
&H^{|\Lambda|}\big(\bigcap_{I \in \Lambda} \pi_1^*(I \otimes L^s) \otimes \pi_2^*(\widehat{I} \otimes L^s)\big) \\
&= H^1\Big(\pi_1^*\big((H^0(L^r) \otimes \mathcal{O}_S)^{\otimes p+1} \otimes L^s\big) \otimes \pi_2^*\big(M_{L^r}^{\otimes p+1} \otimes L^s\big)\Big) \\
&= \big(H^0(L^r)^{\otimes p+1} \otimes \underbrace{H^1(L^s)}_{=0}\big) \otimes H^0(M_{L^r}^{\otimes p+1} \otimes L^s) \\
&\quad \oplus \big(H^0(L^r)^{\otimes p+1} \otimes H^0(L^s)\big) \otimes \underbrace{H^1(M_{L^r}^{\otimes p+1} \otimes L^s)}_{=0} \\
&= 0
\end{aligned}
$$

(iii) $\nu = 0$:
In this case we get $\mu = p+1$ and obtain $A = M_{L^r}^{\otimes p+1}$. This means that for all $j = 1 \dots p+1$, there exits a $I_1 \otimes \dots \otimes I_{p+1}$ with $I_j = M_{L^r}$. Again we distinguish two cases:

a) The set Λ consists of only one element: $\Lambda = \{M_{L^r}^{\otimes p+1}\}$.
Here we can do a calculation similar to the previous case.

b) If we have more than one element in Λ, there exists an element such that there is a $k \in \{1, \dots, p+1\}$ with $I_k = H^0(L^r) \otimes \mathcal{O}_S$. Therefore, we get $B_k = M_{L^r}$ and $\mu' \geq 1$. Together with equation (4.5) and remark (3.16), we obtain the final vanishing.

q.e.d.

4.6 Corollary. *Let S be a K3 surface and L a globally generated line bundle on S such that $(L.L) \geq 4$. Then, the bundle $(L \boxtimes L)^r$ has property N_p for $r \geq p+1$.*

Proof. By Theorem 3.4, we have to verify $H^1(M_{J^r}^{\otimes p+1} \otimes J^s) = 0$ for all $r, s \geq p+1$. This is true due to the above theorem. q.e.d.

Line bundles on the blowup $\widetilde{S \times S}$

In this section we lift Theorem 4.4 to a blowup.

4.7 Theorem. *Let X be a projective variety and $\varphi : \widetilde{X} \to X$ the blowup along some subvariety, J a globally generated line bundle on X with $H^1(M_{J^r}^{\otimes p+1} \otimes J^s) = 0$ for all $r \geq 2, s \geq p+1$ and some $p \geq 0$. Then for $\mathcal{L} := \varphi^* J$:*

$$H^1(M_{\mathcal{L}^r}^{\otimes p+1} \otimes \mathcal{L}^s) = 0 \;\; \forall r \geq 2, s \geq p+1$$

Proof. From Theorem 2.4 we know that the higher direct images of the structure sheaf of the blowup vanish: $R^i\varphi_*(\mathcal{O}_{\widetilde{X}}) = 0$ for all $i \geq 1$. This holds for $\varphi^*(M_{\mathcal{L}^r}^{\otimes p+1} \otimes \mathcal{L}^s)$, too:

$$\begin{aligned} R^i\varphi_*\big(\varphi^*(M_{\mathcal{L}^r}^{\otimes p+1} \otimes \mathcal{L}^s)\big) &= R^i\varphi_*\big(\varphi^*(M_{\mathcal{L}^r}^{\otimes p+1} \otimes \mathcal{L}^s) \otimes \mathcal{O}_{\widetilde{X}}\big) \\ &= \big(M_{\mathcal{L}^r}^{\otimes p+1} \otimes \mathcal{L}^s\big) \otimes R^i\varphi_*\mathcal{O}_{\widetilde{X}} && \text{projection formula} \\ &= 0. && \forall i \geq 1 \end{aligned}$$

Using the Leray spectral sequence, we get

$$H^1(X, \varphi_*\big(\varphi^*(M_{\mathcal{L}^r}^{\otimes p+1} \otimes \mathcal{L}^s)\big)) = H^1(\widetilde{X}, \varphi^*(M_{\mathcal{L}^r}^{\otimes p+1} \otimes \mathcal{L}^s)). \tag{4.8}$$

Thus, for all $r \geq 2, s \geq p+1$:

$$\begin{aligned} 0 &= H^1(M_{J^r}^{\otimes p+1} \otimes J^s) && \text{by assumption} \\ &= H^1(M_{J^r}^{\otimes p+1} \otimes J^s \otimes \varphi_*\mathcal{O}_{\widetilde{X}}) && \text{Theorem 2.4: } \mathcal{O}_X = \varphi_*\mathcal{O}_{\widetilde{X}} \\ &= H^1(\varphi_*(\varphi^*(M_{J^r}^{\otimes p+1} \otimes J^s) \otimes \mathcal{O}_{\widetilde{X}})) && \text{projection formula} \\ &= H^1(\varphi^*(M_{J^r}^{\otimes p+1} \otimes J^s)) && \text{see formula ((4.8))} \\ &= H^1(M_{\varphi^* J^r}^{\otimes p+1} \otimes \varphi^* J^s) && \text{Corollary 3.21} \\ &= H^1(M_{\mathcal{L}^r}^{\otimes p+1} \otimes \mathcal{L}^s). \end{aligned}$$

q.e.d.

According to the diagram (2.7) we want to blow up the variety $S \times S$ along the diagonal: $\varphi : \widetilde{S \times S} \to S \times S$.

4.9 Corollary. *Let S be a K3 surface and L an ample line bundle on S, generated by global sections with $(L.L) \geq 4$. As before, we define $J := L \boxtimes L$ on $S \times S$ and $\mathcal{L} = \varphi^* J$ on $\widetilde{S \times S}$. Then, $\mathcal{L}^r$ is a normally generated line bundle for all $r \geq 2$ and $\mathcal{L}^s$ satisfies N_p for $s \geq p+1$.*

Proof. To prove the normal generation we use Theorem 3.2 and have to show the vanishing of $H^1(M_{\mathcal{L}^r} \otimes \mathcal{L}^s)$ for all $r, s \geq 2$. We want to apply Theorem 4.7 and need the vanishing of $H^1(M_{J^r} \otimes J^s)$, which was done in the proof of Corollary 4.3.
To prove property N_p we use Theorem 3.4. In order to do so, we firstly need to establish two conditions:

- $H^1(\mathcal{L}^t) = 0 \qquad \forall t \geq 1$:
  The line bundle J^t has vanishing first cohomology because the Künneth formula implies $H^1(J^t) = H^1(L^t) \otimes H^0(L^t) \oplus H^0(L^t) \otimes H^1(L^t)$. The remaining part is similar to the previous proof: $\mathcal{L}$ has vanishing higher direct images, so we can again apply the Leray spectral sequence and perform the same calculations with J instead of $M_{\mathcal{L}^r}^{\otimes p+1} \otimes \mathcal{L}^s$.

- $H^1(M_{\mathcal{L}^r}^{\otimes p+1} \otimes \mathcal{L}^s) = 0 \qquad \forall r, s \geq p+1$:
  Since we are in the situation of Theorem 4.4, we maintain $H^1(M_{J^r}^{\otimes p+1} \otimes J^s) = 0$, to which we can apply the above theorem to obtain the required vanishing. Now, Theorem 3.4 gives us property N_p.

q.e.d.

If one is only interested in the normal generation of $\mathcal{L}^r$, there is a more concrete proof. Using the sequence (3.1) in the usual fashion, we obtain that $H^1(M_{\mathcal{L}^r} \otimes \mathcal{L}^s)$ is the kernel of the multipication map $H^0(\mathcal{L}^r) \otimes H^0(\mathcal{L}^s) \to H^0(\mathcal{L}^{r+s})$ for all r, s. Together with 2.4 and the Leray spectral sequence, we see

$$H^0(\mathcal{L}^t) = H^0(\varphi^* J^t) = H^0(\varphi_* \varphi^* J^t) = H^0(J^t \otimes \varphi_* \mathcal{O}_{\widetilde{S \times S}}) = H^0(J^t) \qquad \forall t.$$

Now, we can write the multiplication map in the following way:

$$\begin{tikzcd}
H^0(\mathcal{L}^r) \otimes H^0(\mathcal{L}^s) \arrow[r] \arrow[d, equal] & H^0(\mathcal{L}^{r+s}) \arrow[d, equal] \\
H^0(J^r) \otimes H^0(J^s) \arrow[r] & H^0(J^{r+s}).
\end{tikzcd}$$

The bottom map is surjective due to the last section (see 4.2). Thus, the top one is.

Line bundles on the Hilbert scheme $\mathrm{Hilb}^2(S)$ and the symmetric power $S^2(S)$

4.10 Theorem. *Let X and Y be two projective varieties, $f : X \to Y$ a morphism and K a globally generated line bundle on Y with the following properties for fixed $r, s \geq 1$:*

(i) The structure sheaf of Y is a direct summand of the direct image of the structure sheaf of X under f.

(ii) The higher direct images $(R^i f_*) f^*(M_{\mathcal{J}^r} \otimes \mathcal{J}^s)$ vanish for all $i \geq 1$.

(iii) The line bundle $J := f^* K$ satisfies $H^1(M_{K^r} \otimes K^s) = 0$.

Then, $H^1(M_{K^r} \otimes K^s) = 0$

Proof. We deploy the first assumption and write $f_*\mathcal{O}_X = \mathcal{O}_Y \oplus I$ for some I. The second assumption implies $H^1(f^*(M_{K^r} \otimes K^s)) = H^1(f_*(f^*(M_{K^r} \otimes K^s)))$ by the Leray spectral sequence. Consequently,

$$\begin{aligned}
0 &= H^1(f^* M_{K^r} \otimes J^s) \\
&= H^1(f_*(f^*(M_{K^r} \otimes K^s))) \\
&= H^1(M_{K^r} \otimes K^s \otimes f_*\mathcal{O}_X) \qquad \text{projection formula (cf. proof of Theorem 4.7)} \\
&= H^1(M_{K^r} \otimes K^s \otimes (\mathcal{O}_Y \oplus I) \\
&= H^1(M_{K^r} \otimes K^s) \oplus H^1(M_{K^r} \otimes K^s \otimes I).
\end{aligned}$$

From this, we deduce that the group $H^1(M_{K^r} \otimes K^s)$ appears as a direct summand of zero. Hence, it must be zero itself. q.e.d.

As in the last sections, we use this theorem to show the normal generation of special line bundles on the symmetric product. This is the statement of the

4.11 Corollary. *Let S be a K3 surface and L an ample line bundle on it, generated by global sections with $(L.L) \geq 4$. The bundle $L \boxtimes L$ descends to a bundle K on the 2nd symmetric power $S^2(S)$. The bundle K^r is normally generated for $r \geq 2$.*

Proof. We want to use Theorem 3.2 again and show $H^1(M_{K^r} \otimes K^s) = 0$ for all $r, s \geq 2$. Theorem 4.10 gives us this vanishing, but we have to check several conditions:

(i) This hypothesis is satisfied as mentioned in (2.8).

(ii) The vanishing of the higher direct images can also be deduced from (2.8) using the projection formula as done several times before.

(iii) We are in the same situation as in Lemma 3.17, which yields exactly the desired vanishing.

q.e.d.

4.12 Corollary. *Let S be a K3 surface and L an line bundle on it, generated by global sections with $(L.L) \geq 4$. Let $\mathcal{L}$ be the pullback of the line bundle $\mathcal{L} \boxtimes L$ to the blowup along the diagonal. This bundle descends to a bundle $\mathcal{J}$ on the Hilbert scheme* $\mathrm{Hilb}^2(S)$. *The bundle $\mathcal{J}^r$ is normally generated for $r \geq 2$.*

Proof. This proof is almost identical to the one above: show $H^1(M_{\mathcal{J}^r} \otimes \mathcal{J}^s) = 0$ for all $r, s \geq 2$ and use Theorem 3.2. For this vanishing we use Theorem 4.10. The first two conditions are true for the same reason (except one uses (2.9)) and the third one is satisfied due to Lemma 3.19. q.e.d.

Perspectives

Now, we have proven the theorems mentioned in the introduction. Given a line bundle L on a K3 surface, we have proven that L^r is normally generated for $r \geq 2$ and satisfies property N_p if $r \geq p+1$. The same we have proven to be true for the bundle $L \boxtimes L$ on the self-product of a K3 surface as well as the pullback of this bundle on the blowup.
However, looking at the Hilbert scheme or the symmetric product we have only shown that the bundle obtained by descent has a normally generated square. The next question is, of course, whether this bundle, or its tensor powers, satisfies property N_p. Therefor, it would be necessary to show the surjectivity of the map

$$H^0(M_{K^r}^{\otimes p+1} \otimes K^s) \otimes H^0(K) \to H^0(M_{K^r}^{\otimes p+1} \otimes K^{s+1})$$

for $r \geq 2$ and $s \geq p+1$. Now, there are two natural generalizations.
Firstly, one could ask for these theorems to be true on the n-th self-product of a K3 surface, the Hilbert scheme of n points on a K3 surface, etc. Since the kernel of the evaluation map acts well, we hope to be able to generalize these theorems to objects of this kind.
The second question(,) that one has in mind, is more interesting: What about arbitrary line bundles on these varieties? Do we need to go to higher tensor powers or is it enough to go to the square, if one wants the normal generation? Gallego and Purnaprajna stated a meta-principal:
If L is the product of $(p+1)$ ample and base-point-free line bundles satisfying "certain" cohomological and numerical conditions, then L satisfies the condition N_p.
Believing in this, as this thesis suggests, the $p+1$-th power of a line bundle should have property N_p, as long as the bundle is "nice".

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