Exercise-sheet 10 (January 11, 2015)

1 In-class exercises

In this session we will revisit some of the equilibrium properties of Landau Fermi-liquids discussed in the lecture. First though, what are the assumptions behind Landau's theory?

1.1 Quasi-particle distribution in Landau's theory

In a macroscopic state of thermal equilibrium, the smoothed distribution function $n_{\boldsymbol{p}\sigma}$ may be determined from the fact that for *any* variation about thermodynamic equilibrium at finite temperature

$$\delta E = T\delta s + \mu\delta n, \tag{1}$$

where E is the energy (per unit volume) of the system, δs is the variation of the entropy density, δn is the variation of the particle density, T is the temperature and μ is the chemical potential.

Given that

$$n = \frac{1}{V}\sum_{\boldsymbol{p}\sigma} n_{\boldsymbol{p}\sigma} \quad \text{and} \quad s = -\frac{1}{V}\sum_{\boldsymbol{p}\sigma}[n_{\boldsymbol{p}\sigma}\ln n_{\boldsymbol{p}\sigma} + (1-n_{\boldsymbol{p}\sigma})\ln(1-n_{\boldsymbol{p}\sigma})], \tag{2}$$

(I) show

$$n_{\boldsymbol{p}\sigma} = \frac{1}{e^{\beta(\epsilon_{\boldsymbol{p}\sigma}-\mu)}+1}, \tag{3}$$

where $\epsilon_{\boldsymbol{p}\sigma} = \epsilon_{\boldsymbol{p}\sigma}[n_{\boldsymbol{p}\sigma}]$.

(II) For slight perturbations about zero temperature equilibrium, the quasi-particle distribution function varies only in the neighbourhood of the Fermi surface. Consider, then, the state produced by adding a quasi-particle to the ground-state. Its energy, measured relatively to the ground-state is given by $\epsilon^{(0)}_{\boldsymbol{p}\sigma} = \epsilon_{\boldsymbol{p}\sigma}[n^{(0)}_{\boldsymbol{p}\sigma}]$, where the superscript (0) denotes the ground state.

In the neighbourhood of the Fermi surface we then write

$$\epsilon^{(0)}_{\boldsymbol{p}\sigma} = \mu + v_f(p - p_f), \tag{4}$$

where $v_f = \left(\dfrac{\partial \epsilon^{(0)}_{p\sigma}}{\partial p}\right)_{p=p_f}$. Show the density of quasi-particles at the Fermi surface is

$$N(0) = \frac{m^* p_f}{\pi^2}, \tag{5}$$

where m^* is the effective mass defined as $p_f = m^* v_f$.

1.2 Entropy and specific heat

(I) Use eqs. (2) and (3) to show that under variations of the temperature

$$\delta s = \frac{1}{TV} \sum_{\boldsymbol{p}\sigma} (\epsilon_{\boldsymbol{p}\sigma} - \mu) \delta n_{\boldsymbol{p}\sigma}, \tag{6}$$

and

$$\delta n_{\boldsymbol{p}\sigma} = \frac{\partial n_{\boldsymbol{p}\sigma}}{\partial \epsilon_{\boldsymbol{p}\sigma}} \left(-\frac{\epsilon_{\boldsymbol{p}\sigma} - \mu}{T} \delta T + \delta \epsilon_{\boldsymbol{p}\sigma} - \delta\mu \right), \tag{7}$$

and as a consequence the first contribution to the entropy is due to the explicit δT and is

$$\delta s = -\frac{1}{V} \sum_{\boldsymbol{p}\sigma} \frac{\partial n_{\boldsymbol{p}\sigma}}{\partial \epsilon_{\boldsymbol{p}\sigma}} (\epsilon_{\boldsymbol{p}\sigma} - \mu)^2 \frac{\delta T}{T^2}. \tag{8}$$

(II) Replace the sum by an integral over energy and show that

$$s = \frac{\pi^2}{3} N(0) T, \tag{9}$$

and that the specific heat

$$c_V = T \left(\frac{\partial s}{\partial T} \right)_V = s = \frac{m^* p_f}{3} T, \tag{10}$$

2 Homework - due date: January 18, 2016 (25 points).

2.1 Sommerfeld expansion

Consider the following integral

$$I = \int_0^\infty \frac{f(\epsilon) d\epsilon}{e^{\beta(\epsilon - \mu)} + 1}. \tag{11}$$

where $f(\epsilon)$ is such that I converges and $\beta = 1/T$

(a) Perform the change of variables $z = \beta(\epsilon - \mu)$ and show that the integral (11) can be rewritten as

$$I = T\int_0^{\mu/T} \frac{f(\mu - Tz)}{e^{-z}+1}\,dz + T\int_0^{\infty} \frac{f(\mu + Tz)}{e^{z}+1}\,dz. \tag{12}$$

(b) Further, rewrite I as follows

$$I = T\int_0^{\mu/T} f(\mu - Tz)\,dz - T\int_0^{\mu/T} \frac{f(\mu - Tz)}{e^{z}+1}\,dz + T\int_0^{\infty} \frac{f(\mu + Tz)}{e^{z}+1}\,dz. \tag{13}$$

After replacing the upper limit in the second integral above by ∞ (such a replacement makes sense at temperatures for which $\mu/T \gg 1$ and thus amount to neglecting exponentially small terms) one gets

$$I = \int_0^{\mu} f(\epsilon)\,d\epsilon + T\int_0^{\infty} \frac{f(\mu + Tz) - f(\mu - Tz)}{e^{z}+1}\,dz. \tag{14}$$

(c) Expand the numerator of the integrand in the equation above as a Taylor series of z to get

$$I = \int_0^{\mu} f(\epsilon)\,d\epsilon + 2T^2 f'(\mu)\int_0^{\infty} \frac{z}{e^{z}+1}\,dz + O(T^4). \tag{15}$$

(d) (BONUS+5) Show the integral

$$I_2 = \int_0^{\infty} \frac{z^{x-1}dz}{e^{z}+1} = \int_0^{\infty} z^{x-1}\sum_{n=0}^{\infty}(-1)^n e^{-(n+1)z}dz, \tag{16}$$

and perform the change of variables $y = nz$ to write

$$I_2 = (1 - 2^{1-x})\Gamma(x)\zeta(x), \tag{17}$$

where $\Gamma(x)$ is the gamma function and $\zeta(x)$ the Riemann zeta function.

Comment: The result (17) allows to perform the integrals in equation (15) to yield the final expression

$$I = \int_0^{\mu} f(\epsilon)\,d\epsilon + \frac{\pi^2}{6}T^2 f'(\mu) + O(T^4). \tag{18}$$

2.2 Entropy and specific heat of the free electron gas

Given the expression for the Free energy

$$F = -T\sum_{\boldsymbol{p}\sigma} \ln(1 + e^{\beta(\mu - \epsilon_{\boldsymbol{p}\sigma})}) \tag{19}$$

(a) Show that replacing the sum in equation (19) by and integral over energy and integrating by parts yields

$$F = -\frac{2}{3}\frac{V}{\pi^2}2^{1/2}m^{3/2}\int_0^\infty \frac{\epsilon^{3/2}d\epsilon}{e^{\beta(\epsilon-\mu)}+1} \tag{20}$$

(b) Use Sommerfeld's expansion to show

$$F = F_0 - \frac{VT^2\sqrt{2\mu}m^{3/2}}{6}, \tag{21}$$

where F_0 denotes the value of F at absolute zero.

(c) At low temperatures we can approximate the chemical potential by $\mu = \frac{p_f^2}{2m}$. Use eq. (21) to calculate the entropy and specific heat for the non-interacting electron gas. How do your results compare with the ones obtained within Landau's Fermi-liquid theory?